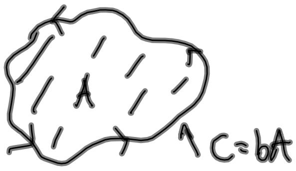
Avslutter Green's Teorem ("Bevis").

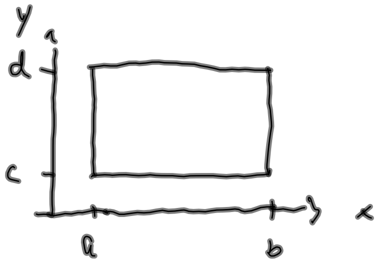
Viktig: P og Q er kont. deriverbare på et område som inneholder A.



$$\int_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy .$$

Ligner på: $\int_a^b f'(x)\,dx = f(b) - f(a)$

Bevis for Green når A er rektangel:



$A = [a,b] \times [c,d]$

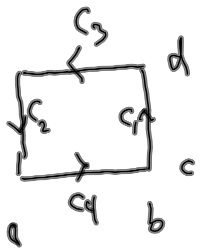
$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_A \frac{\partial Q}{\partial x}\,dx\,dy - \iint_A \frac{\partial P}{\partial y}\,dx\,dy$$

$$= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x}\,dx\right) dy - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y}\,dy\right) dx$$

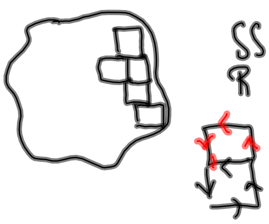
A-f.thm

$$= \int_c^d Q(b,y)\,dy - \int_c^d Q(a,y)\,dy - \left(\int_a^b P(x,d)\,dx - \int_a^b P(x,c)\,dx\right)$$



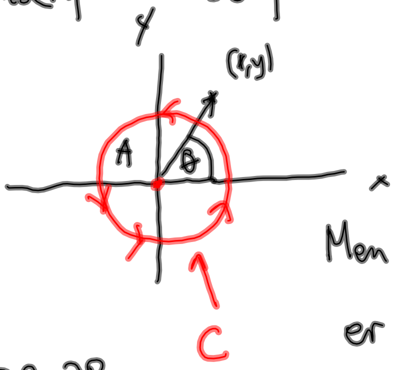
$$\int_{C_1} Q\,dy + \int_{C_2} Q\,dy + \int_{C_3} P\,dx + \int_{C_4} P\,dx$$

$$= \int_C P\,dx + Q\,dy .$$



$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \int_{bR} P\,dx + Q\,dy .$$

Eksempel: Ser på $\mathbb{R}^2 \setminus \{0\}$



Def: $\phi(x,y) := \arg((x,y))$

Kun veldef mod $2k\pi$, $k \in \mathbb{Z}$.

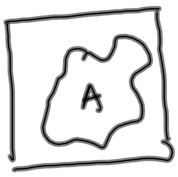
Men $(P,Q) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)$ er veldef !!

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \neq \int_C P\,dx + Q\,dy = 2\pi .$$

$$\text{Men } \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0 .$$

Jordan-målbare mengder

DEF 6.6.1 Vi sier at en begrenset mengde A er Jordan-målbar dersom $\iint_A 1_A \, dx\,dy$ eksisterer.



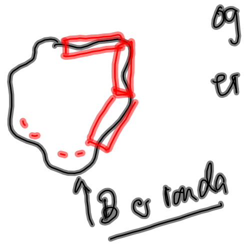
$$1_A(x,y) = \begin{cases} 1 & \text{dersom } (x,y) \in A \\ 0 & \text{ellers.} \end{cases}$$

DEF 6.6.2 En begrenset mengde $B \subset \mathbb{R}^2$ har innhold 0 dersom det for enhver $\varepsilon > 0$ eksisterer rektangler

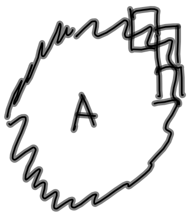
$$R_j = [a_j, b_j] \times [c_j, d_j], \quad j = 1, \dots, m,$$

slik at $B \subset R_1 \cup R_2 \cup \dots \cup R_m$,

og slik at summen av arealene til R_j'ene er mindre enn ε.



Teorem 6.6.3 En begrenset mengde A er Jordan-målbar hvis og bare hvis randa bA har innhold 0.



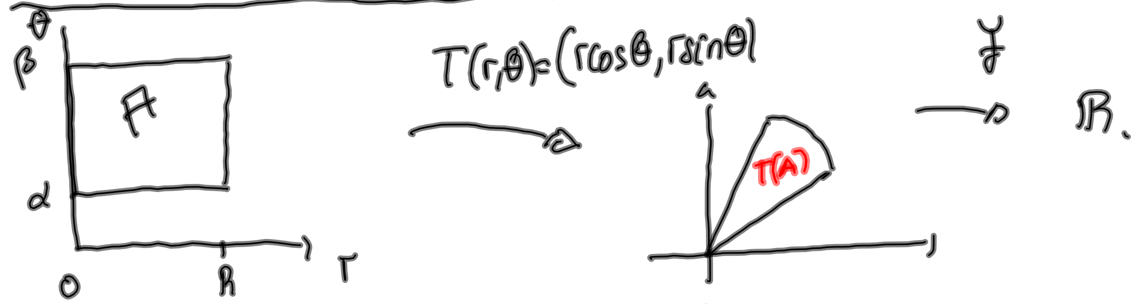
$$\iint_A 1_A\,dx\,dy.$$

$$\overline{\iint_A} 1_A\,dx\,dy = \underline{\iint_A} 1_A\,dx\,dy.$$

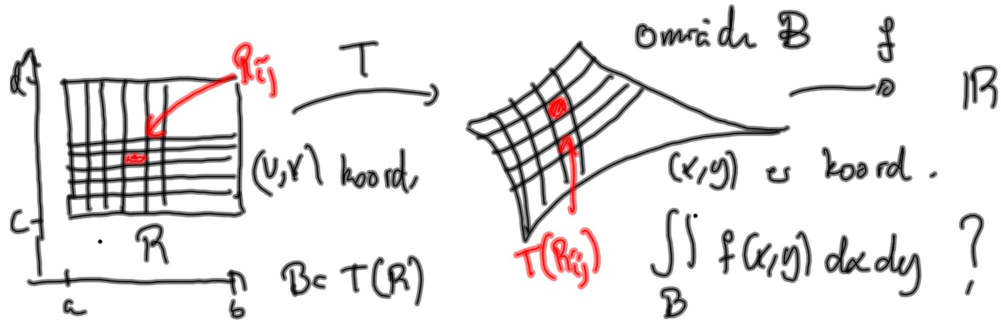
Teorem 6.6.6: La A være en lukket begrenset Jordan-målbar mengde i $\mathbb{R}^2$. Da er enhver kont. funksjon f på A integrerbar over A.

Setning: Type I- og Type II-områder er Jordan-målbare.

Skifte av variable i dobbeltintegraler



$$\iint_{T(A)} f\,dx\,dy = \iint_A f(T(r,\theta))\cdot r\,dr\,d\theta.$$



$$\iint_B f(x,y)\,dx\,dy \;?$$

Kan vi relatere $\iint_B f(x,y)\,dx\,dy$ til et integral som inneholder $f(T(u,v))$?

Velg partisjon av R: $a = x_0 < x_1 < \dots < x_n = b$
$c = y_0 < y_1 < \dots < y_m = d$
velg $c_{ij} \in R_{ij}$.

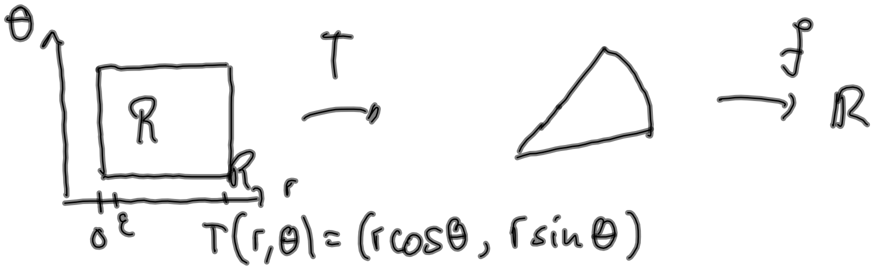
$$\iint_B f(x,y)\,dx\,dy \approx \sum_{i,j} f(T(c_{ij}))\cdot(\text{arealet til } T(R_{ij}))$$

$$\approx \sum_{i,j} f(T(c_{ij}))\cdot|\det T'(c_{ij})|$$

Det siste uttrykket er en Riemannsum for integralet
$$\iint_R f(T(u,v))\cdot|\det T'(u,v)|\,du\,dv$$

<u>Teorem 6.7.1</u>: La U være en åpen mengde i $\mathbb{R}^2$, og anta at $T: U \to \mathbb{R}^2$ er en injektiv avbilding med kont. partiellderiverte, og $\det T' \neq 0$ på hele U. Hvis $D \subset U$ er en lukket og begrenset Jordan-målbar mengde, og $f: D \to \mathbb{R}$ kontinuerlig, så er
$$\iint_{T(D)} f(x,y)\,dx\,dy = \iint_D f(T(u,v))\,|\det T'(u,v)|\,du\,dv.$$

Eks: Polarkoordinater



$T(r,\theta) = (r\cos\theta, r\sin\theta)$

$$\iint_{T(R)} f(x,y)\,dx\,dy = \iint_R f(T(r,\theta))\cdot r\,dr\,d\theta.$$

Ved forrige resultat: $T'(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$

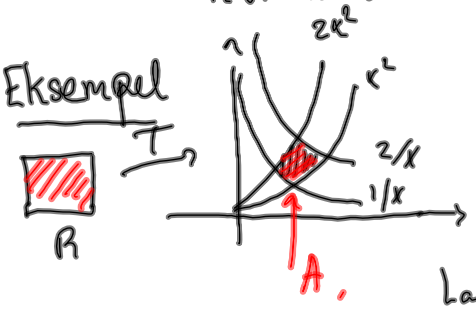
$$\det T'(r,\theta) = r\cos^2\theta + r\sin^2\theta = r,$$

La $\varepsilon > 0$ være liten, og set $R_\varepsilon = [\varepsilon, R]\times[\alpha,\beta]$.

$$\iint_{T(R_\varepsilon)} f(x,y)\,dx\,dy = \iint_{R_\varepsilon} f(T(u,v))\cdot r\cdot du\,dv.$$

Kan nå ta en grense når $\varepsilon \to 0$,

Eksempel



La A være området s.a.

(i) $x^2 \leq y \leq 2x^2$

(ii) $1/x \leq y \leq 2/x$.

La $f(x,y) = y^3$,

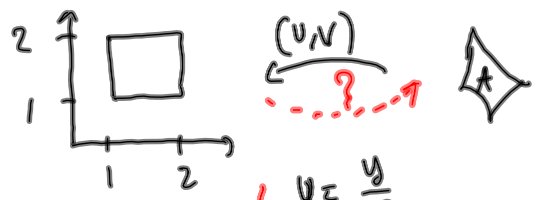
$$\iint_A f(x,y)\,dx\,dy \ ?$$

Skritt 1: Finn avbilding fra A til R.

Skriv om (i) og (ii) (i) $1 \leq y/x^2 \leq 2$

(ii) $1 \leq x\cdot y \leq 2$.

Dermed avbilder funksjonen $(u,v) := (y/x^2, x\cdot y)$ området A på rektangelet $[1,2]\times[1,2]$,



$$\begin{cases} u = \dfrac{y}{x^2} \\ v = xy \,. \end{cases}$$

Steg 2: Finn den inverse til avbildingen (u,v).

$y = u \cdot x^2 \qquad v x^2 = \dfrac{v}{x}$

$y = \dfrac{v}{x} \qquad x^3 = \dfrac{v}{u}$

$$\boxed{x = \left(\frac{v}{u}\right)^{1/3} = v^{1/3} \cdot u^{-1/3}}$$

$x^2 = \dfrac{y}{u} \qquad \dfrac{y}{u} = \dfrac{v^2}{y^2}$

$x^2 = \left(\dfrac{v}{y}\right)^2 \qquad y^3 = v^2 \cdot u$

$$\boxed{y = v^{2/3} \cdot u^{1/3}}$$

Så vi setter $T(u,v) := \left(v^{1/3} u^{-1/3},\ v^{2/3} \cdot u^{1/3}\right) = \left(x(u,v), y(u,v)\right)$

Steg 3: Regn ut Jacobideterminant.

$\dfrac{\partial x}{\partial u} = -\frac{1}{3} u^{-4/3} \cdot v^{1/3}$

$$T'(u,v) = \begin{pmatrix} -\frac{1}{3} u^{-4/3} v^{1/3} & \frac{1}{3} v^{-2/3} u^{-1/3} \\ \frac{1}{3} u^{-2/3} v^{2/3} & \frac{2}{3} v^{-1/3} u^{1/3} \end{pmatrix}$$

$\dfrac{\partial x}{\partial v} = \frac{1}{3} v^{-2/3} \cdot u^{-1/3}$

$\dfrac{\partial y}{\partial u} = \frac{1}{3} u^{-2/3} \cdot v^{2/3}$

$\dfrac{\partial y}{\partial v} = \frac{2}{3} v^{-1/3} u^{1/3}$

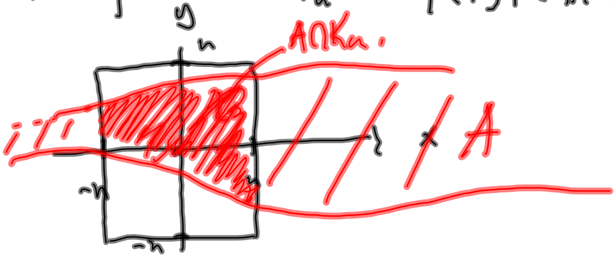
$$\left|\det(T'(u,v))\right| = \left|-\frac{2}{9} u^{-1} - \frac{1}{9} u^{-1}\right| = \left|-\frac{3}{9} u^{-1}\right| = \left|-\frac{1}{3} u^{-1}\right| = \frac{1}{3} u^{-1}.$$

$$\iint_A y^3 \,dx\,dy = \int_1^2 \int_1^2 \left(u^{1/3} v^{2/3}\right)^3 \cdot \frac{1}{3} u^{-1} \,du\,dv = \frac{1}{3} \int_1^2 \int_1^2 v^2 \,du\,dv$$

$= \underline{\text{on}}$.

Uegentlige integraler i planet

Notasjon: $K_n := \{(x,y) \in \mathbb{R}^2 : |x|, |y| \leq n\}$.



DEF 6.8.1: La $A \subset \mathbb{R}^2$ være en delmengde av $\mathbb{R}^2$ s.a. $A \cap K_n$ er Jordan-målbar for alle $n \in \mathbb{N}$. Dersom $f : A \to \mathbb{R}$ er en ikke-negativ kont. funksjon, definerer vi

$$\iint_A f(x,y)\,dx\,dy = \lim_{n \to \infty} \iint_{A \cap K_n} f(x,y)\,dx\,dy$$

dersom grensen eksisterer. I så fall sier vi at integralet konvergerer; ellers sier vi at det divergerer.