Plenum 19/2

3.4: 4, (7), 8, 12, (15 a-c)

3.5: 11

3.6: 9

3.4: Linjeintegraler for vektorfelt

4.) $\vec{F}(x,y,z) = \frac{z}{x}\vec{i} + y\vec{j} + x\vec{k}$

$\mathcal{C}: \vec{r}(t) = e^t\vec{i} + \ln(t)\vec{j} + \sin(t)\vec{k}, \quad t \in [1,2]$

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_1^2 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_1^2 \left(\frac{\sin t}{e^t}, \ln(t), e^t\right)\cdot\left(e^t, \frac{1}{t}, \cos t\right)dt$$

$$= \int_1^2 \left(\sin t + \frac{\ln t}{t} + e^t\cos t\right)dt$$

$$= \left[-\cos t\right]_{t=1}^{2} + \int_1^2 \frac{\ln t}{t}\,dt + \int_1^2 e^t\cos t\,dt$$

M1: $\int_1^2 \frac{\ln t}{t}\,dt = \int_0^{\ln 2} u\,du = \left[\frac{1}{2}u^2\right]_{u=0}^{\ln 2}$

$u = \ln t$
$du = \frac{1}{t}\,dt$

$t=1 \Rightarrow u = 0$
$t=2 \Rightarrow u = \ln 2$

$= \frac{1}{2}(\ln 2)^2$

M2: $\int \overset{u}{e^t}\,\overset{v'}{\cos t}\,dt = e^t \sin t - \int e^t \sin t\,dt$

$v = \sin t$
$u' = e^t$

$= e^t \sin t - \left(e^t(-\cos t) + \int e^t \cos t\,dt\right)$

$= e^t \sin t + e^t \cos t - \int e^t \cos t\,dt$

$2\int e^t \cos t\,dt = e^t(\cos t + \sin t)$

$\int e^t \cos t\,dt = \frac{e^t}{2}(\cos t + \sin t)$

$\Downarrow$

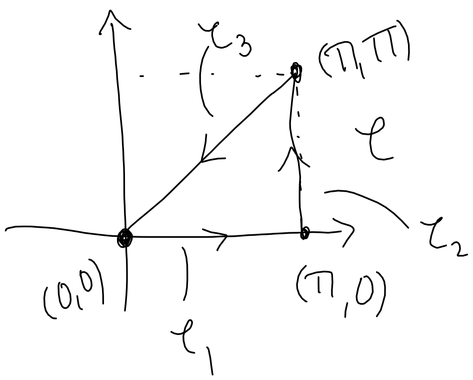
$\int_1^2 e^t \cos t\,dt = \left[\frac{e^t}{2}(\cos t + \sin t)\right]_{t=1}^2$

$= \frac{e^2}{2}(\cos 2 + \sin 2) - \frac{e}{2}(\cos 1 + \sin 1)$

Så:

$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \cos 1 - \cos 2 + \frac{1}{2}(\ln 2)^2$
$+ \frac{e^2}{2}(\sin 2 + \cos 2) - \frac{e}{2}(\cos 1 + \sin 1)$

8) $\vec{F}(x,y,z) = (\cos x \sin y, x)$



$\mathcal{C}$: Kan parametriseres vha 3 kurver:

$\mathcal{C}_1: \vec{r}_1(t) = (t, 0),\ t \in [0,\pi]$

$\mathcal{C}_2: \vec{r}_2(t) = (\pi, t),\ t \in [0,\pi]$

$\mathcal{C}_3: \vec{r}_3(t) = (\pi - t, \pi - t),\ t \in [0,\pi]$

$\vec{F}(\vec{r}_1(t)) = \underline{(0, t)}$

$\vec{F}(\vec{r}_2(t)) = (\underline{-\sin t, \pi})$

$\vec{F}(\vec{r}_3(t)) = (\cos(\pi - t)\sin(\pi - t), \pi - t)$

$= (\underline{-\cos t \sin t, \pi} - t)$

sin & cos til sum av vinkler

$\vec{r}_1{}'(t) = (1, 0)$

$\vec{r}_2{}'(t) = (0, 1)$

$\vec{r}_3{}'(t) = (-1, -1)$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \underbrace{\int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r}_1}_{\int_0^{\pi} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\, dt} + \int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r}_2 + \int_{\mathcal{C}_3} \vec{F} \cdot d\vec{r}_3$$

helt tilsv.

$$= \int_0^{\pi} \{ 0 + \cancel{\pi} + \cos t \sin t - \cancel{\pi} + t \} \, dt$$

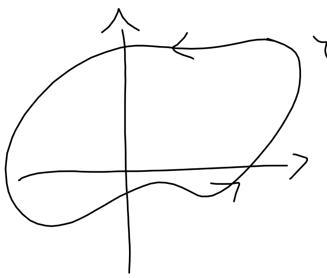
$$= \int_0^{\pi} \{ t + \tfrac{1}{2} \sin(2t) \} \, dt$$

Trig. formel: $\sin(2t)$

$$= \left[\tfrac{1}{2} t^2 - \tfrac{1}{4} \cos(2t) \right]_{t=0}^{\pi}$$

$$= -\tfrac{1}{4} + \frac{\pi^2}{2} + \tfrac{1}{4} = \underline{\underline{\frac{\pi^2}{2}}}$$

12.) $\mathcal{C}$ lukket kurve.



Pf: Anta at vi har to parametriseringer av $\mathcal{C}$:

$$\vec{r}_1(t), \quad t \in [T_1, T_2]$$

$$\vec{r}_2(s), \quad s \in [S_1, S_2].$$

med samme orientering.

Da fins det en funksjon som "slynger s opp på t":

$$s = \frac{\Delta S}{\Delta T}(t - t_0) + s_0 = \underbrace{\frac{S_2 - S_1}{T_2 - T_1}(t - T_1) + S_1}_{\text{en kont. \& deriverbar (kont.) funk. i } t}$$

ettpkt. formel

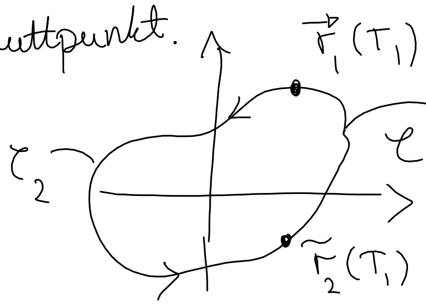
Da er: $\vec{r}_2(s)$ ekvivalent med

$$\tilde{r}_2(s) := \vec{r}_2\left(\frac{S_2-S_1}{T_2-T_1}(t-T_1)+S_1\right), \quad t\in[T_1,T_2]$$

Fra Set. 3.4.4:

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}_2 = \int_{\mathcal{C}} \vec{F}\cdot d\tilde{r}_2$$

Ser nå på $\tilde{r}_2(t)$ & $\vec{r}_1(t)$, $t\in[T_1,T_2]$. Disse param. samme kurve over samme tid, så eneste mulige forskjell er start og sluttpunkt. $\vec{r}_1(T_1)=\tilde{r}_2(T+c)$



$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}_1 = \int_{\mathcal{C}_2} \vec{F}\cdot d\vec{r}_1 + \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{r}_1$$

$$\int_{\mathcal{C}} \vec{F}\cdot d\tilde{r}_2 = \int_{\mathcal{C}_1} \vec{F}\cdot d\tilde{r}_2 + \int_{\mathcal{C}_2} \vec{F}\cdot d\tilde{r}_2$$

Men: $\int_{\mathcal{C}_2} \vec{F}\cdot d\vec{r}_1 = \int_{\mathcal{C}_2} \vec{F}\cdot d\tilde{r}_2$ siden $\vec{r}_1$ og $\tilde{r}_2$ parametriserer samme kurve over samme tid.

Av samme grunn $\int_{\mathcal{C}_1} \vec{F}\cdot d\vec{r}_1 = \int_{\mathcal{C}_1} \vec{F}\cdot d\tilde{r}_2$

Da er

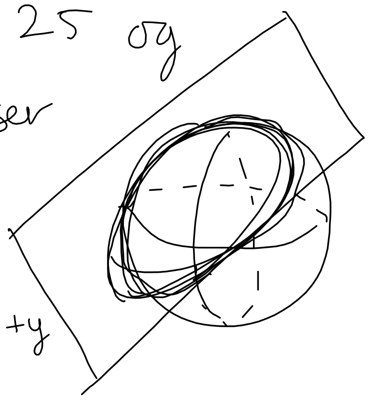
$$\int_C \vec{F}\cdot d\vec{r}_1 = \int_C \vec{F}\cdot d\tilde{r}_2 = \int_C \vec{F}\cdot d\vec{r}_2$$

Dette viser at verdien på linjeint. er uavh. av startpkt. for lukket kurve.

3.5: Gradienter & konservative felt

11) $\vec{F}(x,y,z) = \left(z e^{xz+y},\ e^{xz+y} + 2z,\ x e^{xz+y} + 2y\right)$

C: Skjæring mellom $x^2+y^2+z^2=25$ og $x-2y+3z=1$: Fra figur ser vi at kurven blir lukket.



$$\frac{\partial F_2}{\partial x} = z e^{xz+y}, \quad \frac{\partial F_1}{\partial y} = z e^{xz+y}$$

like!

$$\frac{\partial F_3}{\partial x} = e^{xz+y} + zx e^{xz+y}, \quad \frac{\partial F_1}{\partial z} = e^{xz+y} + xz e^{xz+y}$$

like!

$$\frac{\partial F_2}{\partial z} = x e^{xz+y} + 2, \quad \frac{\partial F_3}{\partial y} = x e^{xz+y} + 2$$

like!

$\Downarrow$

Feltet $\vec{F}$ er konservativt.

Finner potensialfunk. ϕ:

$$\frac{\partial \phi(x,y,z)}{\partial x} = F_1(x,y,z) = z\, e^{xz+y}$$

$$\frac{\partial \phi(x,y,z)}{\partial y} = F_2(x,y,z) = e^{xz+y} + 2z$$

$$\frac{\partial \phi(x,y,z)}{\partial z} = F_3(x,y,z) = x\, e^{xz+y} + 2y$$

$\Downarrow$

$$\phi(x,y,z) = e^{xz+y} + C_1(y,z)$$

$$= e^{xz+y} + 2yz + C_2(x,z)$$

$$= e^{xz+y} + 2zy + C_3(x,y)$$

$\Rightarrow$ $\phi(x,y,z) = e^{xz+y} + 2zy$ er en potensialfunk.

$\Downarrow$

slutt $\mathcal{C}$ start $\mathcal{C}$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}} \nabla\phi \cdot d\vec{r} = \phi(\vec{b}) - \phi(\vec{a})$$

Def. potensial-funk: 3.5.2

Set. 3.5.1

$$= \phi(\vec{a}) - \phi(\vec{a}) = \underline{\underline{0}}$$

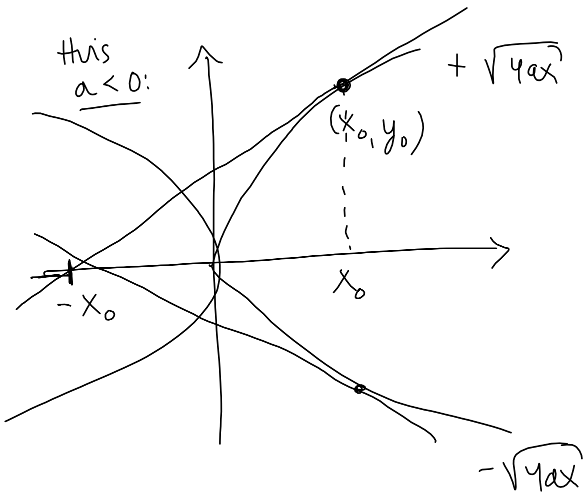
Kurven $\mathcal{C}$ er lukket

3.6: Kjeglesnitt

9.) Vis: Tangeten til parabelen $y^2 = 4ax$ i pkt. (x_0, y_0) skjærer x-aksen i $(-x_0, 0)$.

$$y^2 = 4ax$$

$$y = \pm\sqrt{4ax}$$



Bevis: Anta $y_0 > 0$ og $a > 0$.

Da er: $y_0 = \sqrt{4ax_0}$.

Tangenten i (x_0, y_0) er

$$y = \frac{2a}{\sqrt{4ax_0}}(x - x_0) + \sqrt{4ax_0}$$

ettpunkts-formel

el. def. av tangent

$$\Rightarrow (\sqrt{4ax})' = \left((4ax)^{\frac{1}{2}}\right)' = \frac{1}{2}(4ax)^{-\frac{1}{2}} 4a = \frac{2a}{\sqrt{4ax}}$$

Finner skjæring m/ x-aksen:

$$\frac{2a}{\sqrt{4ax_0}}(x-x_0)+\sqrt{4ax_0}=0$$

$$2a(x-x_0)=-4ax_0$$

$$x-x_0=-2x_0$$

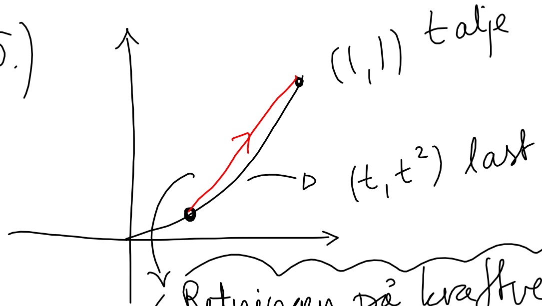
$$\underline{x=-x_0}$$

$\Rightarrow$ Skjæringen med x-aksen er $(-x_0, 0)$.

Helt tilsvarende kan man gjøre for $y_0<0$ og/eller $a<0$. Evt. kan dette sees ved symmetri fra figuren.

3.4:

15.)



$\vec{r}(t)=(t, t^2)$, $t\in[0,1]$

K

Retningen på kraftvektoren som virker på lasten

a) Trekkraften har retning fra lasten mot talja:

$$\underbrace{(1,1)}_{\text{talje}}-\underbrace{(t,t^2)}_{\text{last}}=(1-t, 1-t^2)$$

$$\sqrt{(1-t)^2 + (1-t^2)^2} \underset{\text{3. kvad. set}}{=} \sqrt{(1-t)^2 + (1-t)^2(1+t)^2}$$

$$= (1-t)\sqrt{1+(1+t)^2} = (1-t)\underline{\sqrt{2+2t+t^2}}$$

Enhetsvektor i kraftretningen:

$$\frac{(1-t,\ 1-t^2)}{(1-t)\sqrt{2+2t+t^2}} = \frac{\cancel{1-t}}{\cancel{(1-t)}\sqrt{2+2t+t^2}}(1,\ 1+t)$$

$$= \underline{\frac{1}{\sqrt{2+2t+t^2}}(1,\ 1+t)}$$

Trekkraften er konstant lik K, så kraftvektoren er

$$\vec{K}(t) = \frac{K}{\sqrt{2+2t+t^2}}(1,\ 1+t)$$