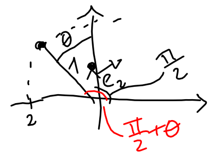
Plenum 4/2-15

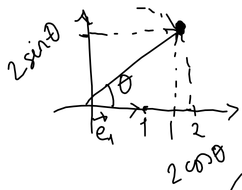
1.9: 6, 10, 11
1.10: 5
2.7: 7, 8, 9
2.8: 2

1.9: Lineæravbildninger

6.) Metode for å finne matrisen til en lineæravbildning:
Se hva avbildningen gjør med enhetsvektorene.

$$\vec{T}(\vec{e}_1) = \vec{T}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\cos\theta\\2\sin\theta\end{bmatrix}$$





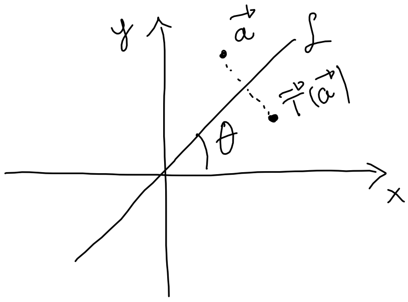
$$\vec{T}(\vec{e}_2) = \vec{T}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\cos(\frac{\pi}{2}+\theta)\\2\sin(\frac{\pi}{2}+\theta)\end{bmatrix}$$

sin/cos sum av vinkler; roterer koordinatsyst.

$$= \begin{bmatrix}-2\sin\theta\\2\cos\theta\end{bmatrix}$$

Matrisen til $\vec{T}$ er $A = \begin{bmatrix}2\cos\theta & -2\sin\theta\\2\sin\theta & 2\cos\theta\end{bmatrix} = 2\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$

10.)



$\vec{T}: \mathbb{R}^2 \to \mathbb{R}^2$; speiler om L

A_ϕ: dreier ϕ i pos. ret.

B: speiler om x-aksen

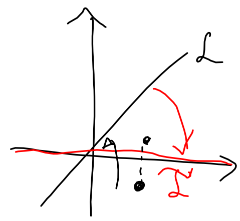
VIS: Matrisen til $\vec{T}$, C, er $C = A_\theta B A_{-\theta}$.

Hvordan speile om $\mathcal{L}$ i flere skritt?

1) Roter $-\theta$ sånn at $\mathcal{L}$ ligger "oppå" x-aksen: $A_{-\theta}$

2) Speil om x-aksen (dvs. roterte $\mathcal{L}$): B

3) Roter θ for å reversere steg 1.): A_θ



Dvs: $C = \underbrace{A_\theta}_{3)} \underbrace{B}_{2)} \underbrace{A_{-\theta}}_{1)}$

$$\begin{aligned} \vec{T}(\vec{x}) &= C\vec{x} \\ &= (A_\theta B A_{-\theta})\vec{x} \\ &= (A_\theta B) A_{-\theta}\vec{x} \end{aligned}$$

$$C = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Eks 3 & Eks. 2

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & \overbrace{\cos\theta\sin\theta + \sin\theta\cos\theta}^{2\sin\theta\cos\theta} \\ \sin\theta\cos\theta + \cos\theta\sin\theta & \underbrace{\sin^2\theta - \cos^2\theta}_{-(\cos^2\theta - \sin^2\theta)} \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

11) $\vec{a} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

a) $\begin{cases} \vec{e}_1 = \underline{x}\vec{a} + \underline{y}\vec{b} \\ \vec{e}_2 = \underline{z}\vec{a} + \underline{u}\vec{b} \end{cases}$

$$\begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x\begin{bmatrix} -2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2x + y \\ x + 3y \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = z\begin{bmatrix} -2 \\ 1 \end{bmatrix} + u\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2z + u \\ z + 3u \end{bmatrix} \end{cases}$$

$\begin{cases} 1 = -2x + y \\ 0 = x + 3y \end{cases} \Rightarrow x = -3y \Rightarrow 6y + y = 1 \Rightarrow y = \underline{\underline{\frac{1}{7}}} \Rightarrow x = -\underline{\underline{\frac{3}{7}}}$

$\begin{cases} 0 = -2z + u \\ 1 = z + 3u \end{cases} \Rightarrow u = 2z \Rightarrow z + 6z = 1 \Rightarrow z = \underline{\underline{\frac{1}{7}}} \Rightarrow u = \underline{\underline{\frac{2}{7}}}$

b) $\vec{T}: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{T}(\vec{a}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{T}(\vec{b}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\vec{T}(\vec{e}_1) = \vec{T}(x\vec{a} + y\vec{b}) = x\vec{T}(\vec{a}) + y\vec{T}(\vec{b})$$

$$= x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} \\ -\frac{3}{7} \end{bmatrix} + \begin{bmatrix} \frac{1}{7} \\ -\frac{1}{7} \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} -\frac{2}{7} \\ -\frac{4}{7} \end{bmatrix}}}$$

a)

$\vec{T}$ er lineær

$$\vec T(\vec e_2) \underset{a)}{=} \vec T(z\vec a + u\vec b) \underset{\vec T \text{ er lineær}}{=} z\vec T(\vec a) + u\vec T(\vec b)$$

$$= \frac{1}{7}\begin{bmatrix}1\\1\end{bmatrix} + \frac{2}{7}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{7}+\frac{2}{7}\\ \frac{1}{7}-\frac{2}{7}\end{bmatrix} = \underline{\underline{\begin{bmatrix}\frac{3}{7}\\ -\frac{1}{7}\end{bmatrix}}}$$

c) Matrisen til $\vec T$:

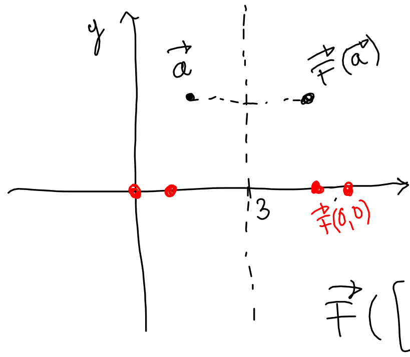
$$\underline{\underline{A = \begin{bmatrix}-\frac{2}{7} & \frac{3}{7}\\ -\frac{4}{7} & -\frac{1}{7}\end{bmatrix}}}$$

1.10: Affinavbildninger

$A\vec x$

$\boxed{A\vec x + \vec b}$

5) a) $\vec F : \mathbb{R}^2 \to \mathbb{R}^2$, affin; Speiler om linja $x = 3$



$\vec F$ affin $\Rightarrow$ $\vec F(\vec x) = A\vec x + \vec b$, $A \in \mathbb{R}^{2\times 2}$, $\vec b \in \mathbb{R}^2$

$$\vec F\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = A\begin{bmatrix}0\\0\end{bmatrix} + \vec b = \vec 0 + \vec b = \underline{\vec b}$$

$$\vec F\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \underline{\begin{bmatrix}6\\0\end{bmatrix}} \Rightarrow \vec b = \begin{bmatrix}6\\0\end{bmatrix}.$$

$A = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$;

$$\vec F\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = A\begin{bmatrix}1\\0\end{bmatrix} + \vec b = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}6\\0\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}\\ a_{21}\end{bmatrix} + \begin{bmatrix}6\\0\end{bmatrix} = \begin{bmatrix}a_{11}+6\\ a_{21}\end{bmatrix}$$

Fra figur:

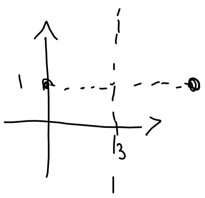
$$\vec{F}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Fra forrige side:

$$\begin{bmatrix}5\\0\end{bmatrix} = \begin{bmatrix}a_{11}+6\\a_{21}\end{bmatrix} \Rightarrow \begin{matrix}a_{11}+6=5\\ \underline{a_{21}=0}\end{matrix} \Rightarrow \underline{a_{11}=-1}$$

Fra figur:

$$\vec{F}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}6\\1\end{bmatrix}$$



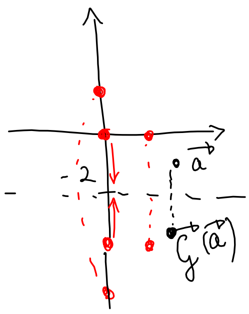
$$\vec{F}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = A\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}6\\0\end{bmatrix} = \begin{bmatrix}a_{12}+6\\a_{22}\end{bmatrix}$$

$$\begin{bmatrix}a_{12}+6\\a_{22}\end{bmatrix} = \begin{bmatrix}6\\1\end{bmatrix} \Rightarrow \begin{matrix}a_{12}=\underline{0}\\ \underline{a_{22}=1}\end{matrix}$$

Så:

$$A = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

b)



$$\vec{G}\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-4\end{bmatrix} \overset{a)}{=} \vec{b}\text{; konstantledd}$$

$$\vec{G}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-4\end{bmatrix}$$

$$\vec{G}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-5\end{bmatrix}$$

Samme regning som i a):

$$\begin{bmatrix}a_{11}+0\\a_{21}+(-4)\end{bmatrix} = \begin{bmatrix}1\\-4\end{bmatrix} \qquad \begin{bmatrix}a_{12}+0\\a_{22}+(-4)\end{bmatrix} = \begin{bmatrix}0\\-5\end{bmatrix} \Rightarrow A = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

2.7: Kjerneregelen

8.) $T = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

$$T = f(r\cos\theta, r\sin\theta)$$

a) $$\frac{\partial T}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x(r,\theta)}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y(r,\theta)}{\partial r}$$

$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

Kjerne-regelen på komponent-form

$$\frac{\partial T}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x(r,\theta)}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y(r,\theta)}{\partial \theta}$$

$$= -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta$$

b) $r = g(t)$, $\theta = h(t)$ $\quad T(r,\theta) \rightsquigarrow T(t)$

$$T'(t) = \frac{\partial T(r,\theta)}{\partial r}\underbrace{\frac{\partial r(t)}{\partial t}}_{=r'(t)} + \frac{\partial T(r,\theta)}{\partial \theta}\underbrace{\frac{\partial \theta(t)}{\partial t}}_{=\theta'(t)}$$

Kjerneregel på komponent-form

$$\overset{a)}{=} \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) g'(t)$$
$$+ \left(-\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta\right) h'(t)$$

9.) Anta: $\exists$ deriverbar. $g : \mathbb{R}^n \to \mathbb{R}$ s.a.

$$f(x_1, x_2, \dots, x_n, g(x_1, x_2, \dots, x_n)) = 0$$

a) La $\vec{G}(x_1, \dots, x_n) = (x_1, \dots, x_n, g(x_1, \dots, x_n))$ og

$$H(x_1, \dots, x_n) = f(\vec{G}(x_1, \dots, x_n)) = 0$$

Da er:

Kjerneregel på matriseform

$$0 = H'(x_1, \dots, x_n) = \left(\frac{\partial f}{\partial u_1}, \dots, \frac{\partial f}{\partial u_{n+1}}\right) \cdot \vec{G}'(x_1, \dots, x_n)$$

$$= \left(\frac{\partial f}{\partial u_1}, \dots, \frac{\partial f}{\partial u_{n+1}}\right) \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ & 0 & & 1 \\ \frac{\partial g}{\partial x_1} & & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix}$$

n × n identitets-matrise

$$= \left(\frac{\partial f}{\partial u_1} + \frac{\partial f}{\partial u_{n+1}} \frac{\partial g}{\partial x_1}, \dots, \frac{\partial f}{\partial u_n} + \frac{\partial f}{\partial u_{n+1}} \frac{\partial g}{\partial x_n}\right)$$

i'te komponent:

$$\frac{\partial f}{\partial u_i} + \frac{\partial f}{\partial u_{n+1}} \frac{\partial g}{\partial x_i}$$

$$\frac{\partial f}{\partial u_i} + \frac{\partial f}{\partial u_{n+1}} \frac{\partial g}{\partial x_i} = 0 \text{ for alle } i = 1, \dots, n$$

$$\frac{\partial g}{\partial x_i} = -\frac{\frac{\partial f}{\partial u_i}}{\frac{\partial f}{\partial u_{n+1}}}$$

Setter inn det aktuelle pkt:

$$\frac{\partial g}{\partial x_i}(x_1,\dots,x_n) = -\frac{\frac{\partial f}{\partial x_i}(x_1,\dots,x_n,g(x_1,\dots,x_n))}{\frac{\partial f}{\partial x_{n+1}}(x_1,\dots,x_n,g(x_1,\dots,x_n))}$$

b) $f(x,y) = x^2+y^2-R^2$ $\quad (f(x,g(x))=0)$

Svarer til a) med $n=1$, så $\vec{G}(x) = (x, g(x))$

$$\frac{\partial f}{\partial x} = 2x \,,\ \frac{\partial f}{\partial y} = 2y \underset{\text{I det aktuelle pkt}}{=} 2g(x)$$

Fra a):

$$g'(x) = \frac{\partial g}{\partial x}(x) = -\frac{\frac{\partial f}{\partial x}(x,g(x))}{\frac{\partial f}{\partial y}(x,g(x))} = -\frac{\cancel{2}x}{\cancel{2}g(x)} = -\frac{x}{g(x)}$$

$$\Downarrow$$

$$\underline{\underline{g'(x) = -\frac{x}{g(x)}}}$$

Geometrisk tolk:

$$f(x, g(x)) = 0$$

$$x^2 + g(x)^2 - R^2 = 0$$

$$x^2 + g(x)^2 = R^2$$

Dvs. at $y = g(x)$ er på sirkelen $x^2 + y^2 = R^2$

Fra uttrykket på forrige side omskrevet:

$$2x + 2g(x)g'(x) = 0$$

$$x + g(x)g'(x) = 0$$

$$\underbrace{(x, g(x))}_{H(x)} \cdot \underbrace{(1, g'(x))}_{H'(x)} = 0$$

punkt på sirkelen

Derivert, dvs. tangenten i pkt.

Dvs: Vektorene som gir pkt'er på sirkelen står vinkelrett på tangenten sin (siden prikkprodukt = 0).

Dette er klart sant for sirkler: