Plenum 4/3-15

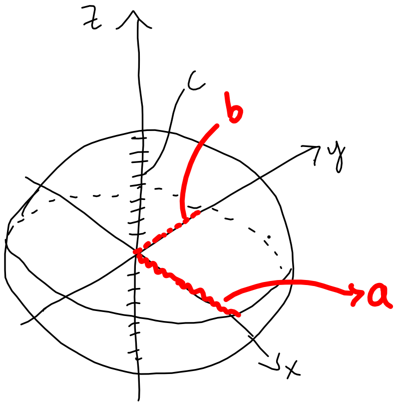
3.9: 6, 8, (14)

6.1: 1efg, 7

6.2: 3

3.9: Parametriserte flater

6) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



Kan omskrives:

$$\underbrace{\left(\frac{x}{a}\right)}_{=\tilde{x}}^2 + \underbrace{\left(\frac{y}{b}\right)}_{=\tilde{y}}^2 + \underbrace{\left(\frac{z}{c}\right)}_{=\tilde{z}}^2 = 1$$

Så: $\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 1$

↳ Kule med sentrum origo, radius 1.

Kulekoordinater: $\tilde{x} = R\sin\phi\cos\theta$

s.195 $\quad \tilde{y} = R\sin\phi\sin\theta$

$\tilde{z} = \underbrace{R}_{=1}\cos\phi \quad , \quad \phi \in [0,\pi], \ \theta \in [0, 2\pi]$

Dvs: $\frac{x}{a} = \tilde{x} = \sin\phi\cos\theta \qquad x = a\sin\phi\cos\theta$

$\frac{y}{b} = \tilde{y} = \sin\phi\sin\theta \Rightarrow y = b\sin\phi\sin\theta$

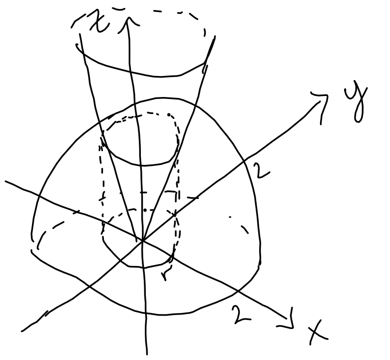
$\frac{z}{c} = \tilde{z} = \cos\phi \qquad z = c\cos\phi$

Parametriseringen er:

$\vec{r}(\phi,\theta) = (a\sin\phi\cos\theta,\ b\sin\phi\sin\theta,\ c\cos\phi),$

$\phi \in [0,\pi]$ & $\theta \in [0, 2\pi]$

8) Kule: $x^2 + y^2 + z^2 = 4 = 2^2$ Kjegle $z^2 = 3(x^2+y^2)$



Skjæring kule & kjegle

$\underbrace{z^2}_{\boxed{\text{kjegle}}} = \underbrace{3(x^2+y^2)}_{\boxed{\text{kula}}} = 3(4-z^2)$

$4z^2 = 12$

$z = \pm\sqrt{3}$

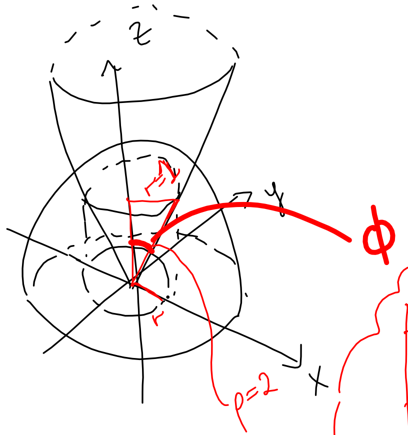
Del over xy-planet $\Rightarrow$ $z = \sqrt{3}$

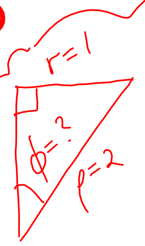
$$4 = x^2 + y^2 + z^2 = r^2 + z^2 = r^2 + 3$$

Kule-koordinater

siden radius ⇒ >0

$$r^2 = 1 \Rightarrow r = \pm 1 \Rightarrow r = 1$$





$$\sin\phi = \frac{1}{2}$$

$$\phi = \frac{\pi}{6}$$

Vet: $\theta \in [0, 2\pi]$ siden vil ha hele sirkelen.

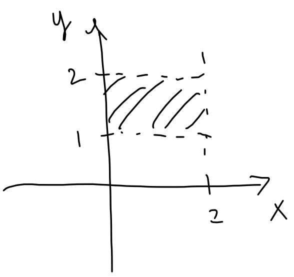
Kulekoordinater:

$$\vec{r}(\theta, \phi) = (2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi)$$

der $\phi \in [0, \frac{\pi}{6}]$, $\theta \in [0, 2\pi]$

6.1: Dobbeltintegraler over rektangler

1.) e) $$\iint_R x y e^{x^2 y}\,dx\,dy = \int_1^2 \int_0^2 x y e^{x^2 y}\,dx\,dy$$



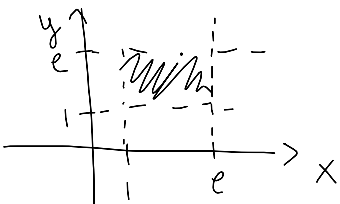
$$= \int_1^2 \left[\frac{1}{2} e^{x^2 y} \right]_{x=0}^{2} dy$$

$$= \int_1^2 \left(\frac{1}{2} e^{4y} - \frac{1}{2} \right) dy$$

$$= \left[\frac{1}{8} e^{4y} - \frac{1}{2} y \right]_{y=1}^{2} = \frac{1}{8} e^8 - 1 - \frac{1}{8} e^4 + \frac{1}{2}$$

$$= \underline{\underline{\frac{1}{8} e^8 - \frac{1}{8} e^4 - \frac{1}{2}}}$$

f) $$\int_1^e \int_1^e \ln(xy)\,dx\,dy = \int_1^e \int_1^e \big(\ln(x) + \ln(y)\big)\,dx\,dy$$



$$= \int_1^e \left\{ \left[x\ln(x) \right]_{x=1}^{e} - \int_1^e 1\,dx + \left[x\ln(y) \right]_{x=1}^{e} \right\} dy$$

Delvis int.
$u' = 1$
$v = \ln x$
$u = x$
$v' = \frac{1}{x}$

$$= \int_1^e \left\{ \cancel{e} - \cancel{e} + 1 + (e-1)\ln(y) \right\} dy$$

$$= \int_1^e \left\{ 1 + (e-1)\ln(y) \right\} dy$$

$$= [y]_{y=1}^{e} + (e-1)\,[y\ln(y)]_{y=1}^{e} - (e-1)\int_1^e 1\,dy$$

$$= e-1+e(e-1)-(e-1)^2$$

$$= 2e-2 = \underline{\underline{2(e-1)}}$$

g) $\displaystyle\int_0^1\int_1^{\sqrt{3}} \frac{1}{1+x^2y}\,dx\,dy =: I$

$\underline{M}$: $\displaystyle\int \frac{1}{1+x^2y}\,dx \underset{\downarrow}{=} \int \frac{1}{\sqrt{y}\,(1+u^2)}\,du$

Substitusjon:
$u = x\sqrt{y}$
$du = \sqrt{y}\,dx$

$$= \frac{1}{\sqrt{y}}\arctan(u) + C = y^{-\frac{1}{2}}\arctan(x\sqrt{y}) + C$$

$$I = \int_0^1 \left[y^{-\frac{1}{2}}\arctan(x\sqrt{y})\right]_{x=1}^{\sqrt{3}} dy$$

$$= \int_0^1 y^{-\frac{1}{2}}\left(\arctan(\sqrt{3}\sqrt{y}) - \arctan(\sqrt{y})\right)dy$$

$$= 2 \int_0^1 \left(\arctan(\sqrt{3}\,u) - \arctan(u) \right) du$$

Substitusjon:

$u = \sqrt{y} = y^{\frac{1}{2}}$

$du = \frac{1}{2} y^{-\frac{1}{2}}\, dy$

$y = 0 \Rightarrow u = 0$

$y = 1 \Rightarrow u = 1$

$$= 2 \left[u \left(\arctan(\sqrt{3}\,u) - \arctan(u) \right) \right]_{u=0}^{1} - 2 \int_0^1 \left(\frac{\sqrt{3}\,u}{1 + 3u^2} - \frac{u}{1+u^2} \right) du$$

Delvis int:

$v' = 1$

$w = \arctan(\sqrt{3}\,u) - \arctan(u)$

$$= 2 \left(\arctan(\sqrt{3}) - \arctan(1) \right) - 2 \left[\frac{\sqrt{3}}{6} \ln(1 + 3u^2) - \frac{1}{2} \ln(1 + u^2) \right]_{u=0}^{1}$$

$$= 2 \left(\frac{\pi}{3} - \frac{\pi}{4} \right) - 2 \frac{\sqrt{3}}{6} \ln(4) + 2 \frac{1}{2} \ln(2)$$

$$= \frac{\pi}{6} + \left(1 - \frac{2\sqrt{3}}{3} \right) \ln(2)$$

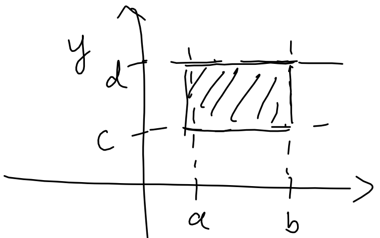
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ikke def.

tan $= \frac{\sin}{\cos}$

7.) Middelverdisetningen for dobbeltintegraler

Anta $f: R \to \mathbb{R}$, kont. funk. $R = [a, b] \times [c, d]$

rektangel



Ekstremalverdiset.

Vis: Fins pkt $(\bar{x}, \bar{y}) \in R$ s.a.

$$\frac{\iint_R f(x,y)\,dx\,dy}{|R|} = f(\bar{x}, \bar{y})$$

arealet til R

Disse eksisterer fordi R er lukket og begrenset og f kontinuerlig

Bevis: La $m := \min_{(x,y)\in R} f(x,y)$ og $M = \max_{(x,y)\in R} f(x,y)$

Da er:

f er mindre enn max

$$\iint_R f(x,y)\,dx\,dy \leq \iint_R M\,dx\,dy$$

$$= M \iint_R 1\,dx\,dy = M\,\underline{|R|}$$

def. av dobbeltint.

f er større enn min

og $$\iint_R f(x,y)\,dx\,dy \geq \iint_R m\,dx\,dy$$

$$= m \iint_R 1\,dx\,dy = \underline{m\,|R|}$$

Så:

$$m\,|R| \le \iint_R f(x,y)\,dx\,dy \le M\,|R|$$

$$\Downarrow \quad (|R|>0; \text{ hvis ikke er det ingenting å vise})$$

$$m \le \frac{\iint_R f(x,y)\,dx\,dy}{|R|} \le M \qquad (\star)$$

Fra skjæringssetningen vet vi at den kont. funk. $f(x,y)$ tar alle verdier mellom minimumet & maksimumet sitt.

m M

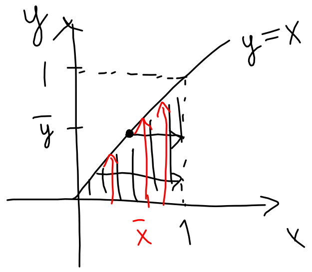
Siden $(\star)$ gir at $\dfrac{\iint_R f(x,y)\,dx\,dy}{|R|}$ er en slik verdi mellom min & maks for f, så må det finnes et punkt $(\bar{x},\bar{y}) \in R$ s.a.

$$f(\bar{x},\bar{y}) = \frac{\iint_R f(x,y)\,dx\,dy}{|R|}$$

$\blacksquare$

6.2: Dobbeltint. over begrensede områder

3) a) $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy =: I$



• = opprinnelig tenkemåte

• = ny tenkemåte

Kan skrives:

$x \in [0,1]$

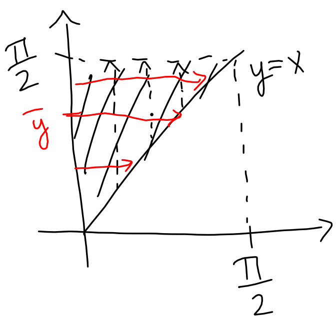
$y \in [0,x]$

$$I = \int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 x e^{x^2}\,dx = \left[\frac{1}{2} e^{x^2}\right]_{x=0}^{1}$$

$$\int_0^1 \left[y e^{x^2}\right]_{y=0}^{x} dx$$

$$= \frac{1}{2}e - \frac{1}{2} = \frac{1}{2}(e-1)$$

b) $\int_0^{\frac{\pi}{2}} \int_x^{\frac{\pi}{2}} \frac{\sin y}{y}\,dy\,dx =: I$



• = oppr. tenkemåte

• = ny — " —

$$I = \int_0^{\frac{\pi}{2}} \int_0^y \frac{\sin y}{y}\,dx\,dy$$

Evt.:

$0 \le x \le \frac{\pi}{2}$

$x \le y \le \frac{\pi}{2}$

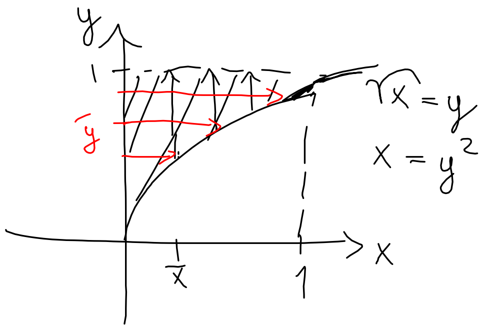
$0 \le x \le y \le \frac{\pi}{2}$

$0 \le y \le \frac{\pi}{2}$

$0 \le x \le y$

$$= \int_0^{\frac{\pi}{2}} \left[x \, \frac{\sin y}{y} \right]_{x=0}^{y} dy = \int_0^{\frac{\pi}{2}} \sin y \, dy$$

$$= \left[-\cos y \right]_{y=0}^{\frac{\pi}{2}} = \underline{\underline{1}}$$



c) $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 e^{\frac{x}{y^2}} \, dy \, dx =: I$

$$\begin{aligned} 0 \le x \le 1 \\ \sqrt{x} \le y \le 1 \end{aligned} \implies 0 \le \sqrt{x} \le y \le 1$$

$$\Downarrow$$

$$\boxed{\begin{aligned} 0 \le y \le 1 \\ 0 \le x \le y^2 \end{aligned}}$$

$$I = \int_0^1 \int_0^{y^2} e^{\frac{x}{y^2}} \, dx \, dy = \dots = \int_0^1 (y^2 e - y^2) \, dy$$

$$= \dots = \underline{\underline{\frac{e-1}{3}}}$$