Divergenstesten: Hvis a_n ikke går mot 0,
$\left(\sum_{n=1}^{\infty} a_n\right)$ så er rekka divergent.

$$\boxed{\begin{array}{l} \sin x \sim x \\ \left(1-\frac{1}{n}\right)^n \to e^{-1} \\ \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a \end{array}}$$

L' Hop.

12.1.4. $\sum_{n=1}^{\infty} \left(\underbrace{1-\sin\frac{1}{n}}\right)^n$

Grense på formen 1^∞. Tar logaritmen!

$$\lim_{n\to\infty} \ln\left(1-\sin\frac{1}{n}\right)^n = \lim \underset{\sim}{n} \ln\left(1-\sin\frac{1}{n}\right) =$$

$$\lim_{n\to\infty} \frac{\ln\left(1-\sin\frac{1}{n}\right)}{\frac{1}{n}} = \lim \frac{-\cos\frac{1}{n}\cdot\cancel{\left(-\frac{1}{n^2}\right)}}{\left(1-\sin\frac{1}{n}\right)\cdot\cancel{\left(-\frac{1}{n^2}\right)}}$$

L'Hop.

$\left(x=\frac{1}{n}, x\to 0\right)$

$$= \lim -\frac{\cos\frac{1}{n}}{\left(1-\sin\frac{1}{n}\right)} = -\frac{\cos 0}{1-\sin 0} = -\frac{1}{1} = -1.$$

$$\lim\left(1-\sin\frac{1}{n}\right)^n = e^{-1} = \frac{1}{e} \neq 0.$$

Divergent ved DT.

12.1.5. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

a) $\displaystyle \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} = \frac{A(k+1)+Bk}{k(k+1)}$$

$$A(k+1) + Bk = \underbrace{(A+B)}_{=0}k + \underbrace{A}_{=1} = 1$$

$A = 1$, $B = -1$.

b) $$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

$$= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots\cdots + \frac{1}{n(n+1)} =$$

$$1 - \cancel{\frac{1}{2}}) + (\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}) + (\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}) + \cdots\cdots + (\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}) + (\cancel{\frac{1}{n}} - \frac{1}{n+1})$$

$$= 1 - \frac{1}{n+1} = s_n \quad \text{Teleskopisk rekke.}$$

c) $\lim s_n = 1$.

dvs $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ er konvergent og $= 1$.

Grensesammenligningstesten GST

Anta $a_n > 0$ og $b_n \geq 0$ for alle n.

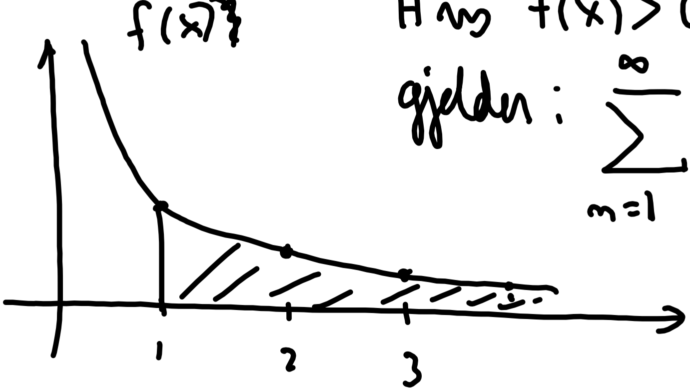
Hvis $\sum_{n=1}^{\infty} a_n$ konvergerer og $\lim \frac{b_n}{a_n} < \infty$, så konvergerer $\sum_{n=1}^{\infty} b_n$.

Hvis $\sum_{1}^{\infty} a_n$ divergerer og $\lim \frac{b_n}{a_n} > 0$, så divergerer $\sum_{1}^{\infty} b_n$.

· Standard sammenligningsrekke $\sum_1 a_n$:

$$\sum_1 \frac{1}{n^p} \quad \text{konvergent} \iff p > 1.$$

· Integraltesten



Hvis $f(x) > 0$ er avtagende, så gjelder: $\sum_{n=1}^{\infty} f(n)$ er konvergent.

$$\Updownarrow$$

$\underbrace{\int_1^{\infty} f(x)\,dx}$ er konvergent..

12.2.1. Bruk integraltesten!

b) $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ konvergent? $\sim \frac{1}{n^2}$.

$$\int_0^{\infty} \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \int_0^R \frac{1}{x^2+1}\,dx =$$

$$\lim_{R\to\infty} \arctan x \Big|_0^R = \lim_{R\to\infty} \arctan R = \underline{\frac{\pi}{2}} < \infty.$$

Altså er $\sum_{0}^{\infty} \frac{1}{n^2+1}$ konvergent ved integraltesten.

12.2.2. $\displaystyle\sum_{n=2}^{\infty} \boxed{\frac{1}{n(\ln n)^p}}$ konvergerer $\Leftrightarrow$ $p > 1$.

Regner ut $\displaystyle\int \frac{dx}{x(\ln x)^p} = \int \frac{1}{u^p}\,du = \begin{cases} \frac{1}{1-p} u^{1-p} & p \neq 1 \\ \ln u & p = 1 \end{cases}$

$\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}}$

$= \begin{cases} \frac{1}{1-p} (\ln x)^{1-p} & p \neq 1 \\ \ln(\ln x) & p = 1 \end{cases}$

Vi ser at $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^p} < \infty \Leftrightarrow p > 1$.

12.2.3 b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n-7}{\underbrace{4n^3+8}_{= b_n}}$ konvergent?

$\dfrac{2n-\cancel{7}}{4n^3\cancel{+8}} \sim \dfrac{1}{2n^2}$. Sammenligner med $\displaystyle\sum_{1}^{\infty} \frac{1}{\underbrace{n^2}_{= a_n}}$, som er konvergent.

$$\lim \frac{b_n}{a_n} = \lim \frac{2n-7}{4n^3+8 \cdot \left(\frac{1}{n^2}\right)} = \lim \frac{2n^3-7n^2}{4n^3+8} = \frac{2}{4} = \frac{1}{2}$$

d) $\sum_{n=1}^{\infty} \frac{\cos\frac{1}{n}}{n+\sqrt{n}}$

$$\frac{\cos\frac{1}{n}}{n+\cancel{\sqrt{n}}} \sim \frac{1}{n}$$

Sammenligna med $\sum^{\infty} \frac{1}{n}$, som er divergent.

$$\lim \frac{b_n}{a_n} = \lim \frac{\cos\frac{1}{n}}{(n+\sqrt{n})\cdot\left(\frac{1}{n}\right)} = \lim \frac{\cos\frac{1}{n}}{1+1/\sqrt{n}} = \frac{\cos 0}{1+0} = 1 > 0$$

altså divergent ved GST.

$$\frac{\left(\frac{\cos\frac{1}{n}}{n+\sqrt{n}}\right)\cdot n}{\left(\frac{1}{n}\right)\cdot n}$$

12.2.5. Forholdstesten FT : $\frac{a_{n+1}}{a_n} \to ?$ <1 konv. >1 div.

Rottesten RT : $\sqrt[n]{a_n} \to ?$ — " —

$=1$ ingen konklusjon

12.2.5

c) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

1^{∞}

$$\lim (a_n)^{\frac{1}{n}} = \lim \left(\left(1-\frac{1}{n}\right)^{n^2}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n} = e^{-1} = \frac{1}{e} < 1$$: Konvergent ved RT.

d) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$. Prøver FT :

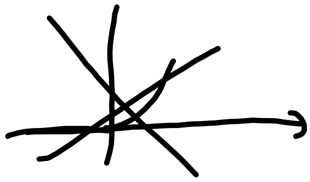
$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\cancel{e^{n+1}} \cdot n!}{(n+1)! \cdot \cancel{e^n}} = \lim_{n\to\infty} \frac{e}{n+1} = 0.$$

Konvergent ved FT.

$$\boxed{\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x\right)}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{n!} = \underline{\underline{e^e - 1.}}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$



g) $\sum_{n=1}^{\infty} \frac{n!\,4^n}{n^n}$.

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\cancel{(n+1)!}\ \cancel{4^{n+1}}}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{n!}\,\cancel{4^n}}$$

$$= \lim \frac{4\cancel{(n+1)}\ n^n}{(n+1)^{n+\cancel{1}}} = \lim \frac{4\,n^n}{(n+1)^n} = \lim \frac{4}{\left(1+\frac{1}{n}\right)^n}$$

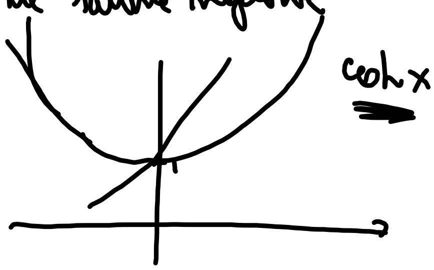
$$= \frac{4}{e} > 1.$$

Divergent ved FT.

12.2.6 c

$$\sum_{n=1}^{\infty} \underbrace{\left(1-\cosh\frac{1}{n}\right)}$$

Alle tallene negative.



Ser på $\sum -a_n$ i stedet:

$$\sum_{n=1}^{\infty} \underbrace{\left(\cosh\frac{1}{n}-1\right)} \sim \frac{1}{n^2} \qquad \left(\cosh x = 1+\frac{x^2}{2}+\dots\right)$$

FT funker ikke:

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\cosh\frac{1}{n+1}-1}{\cosh\frac{1}{n}-1} \overset{\text{L'Hop}}{=} \lim \frac{\left(\sinh\frac{1}{n+1}\right)\cdot\left(-\frac{1}{(n+1)^2}\right)}{\left(\sinh\frac{1}{n}\right)\cdot\left(-\frac{1}{n^2}\right)}$$

$$= \lim \frac{\sinh\frac{1}{n+1}\cdot n^2}{\sinh\frac{1}{n}\cdot(n+1)^2} = \lim \frac{\sinh\frac{1}{n+1}}{\sinh\frac{1}{n}} \overset{\text{L'Hop.}}{=}$$

$$= \lim \frac{\left(\cosh\frac{1}{n+1}\right)\cdot -\frac{1}{(n+1)^2}}{\left(\cosh\frac{1}{n}\right)\left(-\frac{1}{n^2}\right)} = \lim \frac{\left(\cosh\frac{1}{n+1}\right)\cdot n^2}{\left(\cosh\frac{1}{n}\right)(n+1)^2} = 1,$$

dvs. FT gir ikke konklusjon.

Sammenligna med $\sum \frac{1}{n^2}$ som er konvergent.

$$\lim_{n\to\infty} \frac{\cosh\frac{1}{n}-1}{\frac{1}{n^2}} \overset{\text{L'Hop.}}{=} \lim \frac{\left(\sinh\frac{1}{n}\right)\cdot\left(-\frac{1}{n^2}\right)}{-\frac{2}{n^3}} =$$

$$\lim_{n\to\infty} \frac{\sinh\frac{1}{n}}{+\frac{2}{n}} \overset{\text{L'Hop}}{=} \lim \frac{\left(\cosh\frac{1}{n}\right)\cdot\left(-\frac{1}{n^2}\right)}{\left(-\frac{2}{n^2}\right)} = \frac{1}{2} < \infty$$

altså konvergent ved GST.

12.2.7.

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n}$ Sammenligne med $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ som divergerer.

$$\lim \frac{b_n}{a_n} = \cancel{\lim} \frac{\sqrt{n}}{1+n\left(\frac{1}{\sqrt{n}}\right)} = \lim \frac{n}{1+n} = 1 > 0$$

divergent ved GST.

d) $\sum \left(1+\frac{1}{n}\right)^n$ divergent siden $\lim \left(1+\frac{1}{n}\right)^n = e \neq 0$.
ved divergenstesten.

g) $\sum_{n=1}^{\infty} \frac{2^n n!}{(2n)!}$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\cancel{2^{n+1}}\,\cancel{(n+1)!}\,\cancel{(2n)!}}{\cancel{(2n+2)!}\,\cancel{2^n}\cdot\cancel{n!}}$$

$$= \lim \frac{2(n+1)}{(2n+1)(2n+2)} = 0 < 1.$$

Konvergent ved FT.

· En alternerende rekke er konvergent hvis leddene går mot 0 monotont.

12.31

c) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}} = 1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\dots$

er konvergent siden $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$ monotont.

12.41.

a) $\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{n+1}$ er konvergent siden den er alternerende og $\lim_{n\to\infty}\frac{1}{n+1}=0$ monotont.

Men $\sum|a_n| = \sum\frac{1}{n+1}$ er divergent.

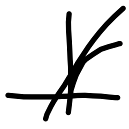
Svar: Betinget konvergent.

b) $\sum_{n=0}^{\infty}(-1)^n \frac{1}{n^2+4}$ er absolutt konvergent siden

$\sum |a_n| = \sum \frac{1}{n^2+4}$ er konvergent $\left(\text{Sammenlign } \sum \frac{1}{n^2}\right)$

GST.

c) $\sum_{n=1}^{\infty}(-1)^n \underbrace{\arcsin\frac{1}{n}}$ er konvergent siden den er alternerende og



$\arcsin x \approx x$

$\frac{1}{\sqrt{1-x^2}}$

$\lim_{n\to\infty} \arcsin \frac{1}{n} = 0$ avtakende,

Absolutt konvergent? Sammenligner $\sum |a_n|$ med $\sum \frac{1}{n}$, som er divergent.

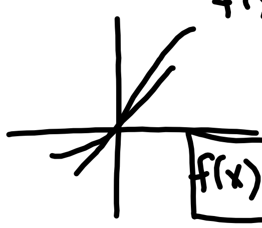
$$\lim_{n\to\infty} \frac{\arcsin\frac{1}{n}}{\frac{1}{n}} \underset{\text{L'Hop}}{=} \lim \frac{\frac{1}{\sqrt{1-(\frac{1}{n})^2}}\cdot\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-\frac{1}{n^2}\right)}} = \lim \frac{1}{\sqrt{1-(\frac{1}{n})^2}} = \underline{1} > 0$$

$\boxed{x = \frac{1}{n}}$

altså divergent ved GST.

$\arcsin x \approx x$

$y\ \boxed{\sin x \approx x} - \frac{1}{6}x^3 + \dots$



$\boxed{f(x) \sim kx}$

$= f'(0)$