12.6. Konvergensintervallet. FT.

1. c) $\sum_{n=0}^{\infty} n \underbrace{(2x-1)}^{n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(n+1)(2x-1)^{n+1}}{n\,\cancel{(2x-1)^{n}}}\right| =$$

$$\lim \frac{n+1}{n}|2x-1| = |2x-1|$$

Konvergent når $|2x-1|<1 \iff |x-\frac{1}{2}|<\frac{1}{2} \iff x\in\underline{(0,1)}$

$x=0$: $\sum(-1)^n n$ divergent ved DT.

$x=1$: $\sum n$ —"—

d) $\sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \left| \frac{(x+1)^{\cancel{n}+1}\sqrt{n}}{\sqrt{n+1}\,\cancel{(x+1)^{n}}} \right| =$$

$$\lim \sqrt{\frac{n}{n+1}}\,|x+1| = |x+1|$$

Konvergent når $|x+1| < 1 \iff x \in (-2, 0)$.

Endepunktene: $x=0$: $\sum_1^{\infty} \frac{1}{\sqrt{n}}$ divergent (siden potensen $\frac{1}{2} < 1$).

$x=-2$: $\sum_1^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ konvergent, alternerende der $a_n \to 0$ monotont.

Konvergensområde: $[-2, 0)$

g) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n \Big\}$

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{\cancel{(n+1)!}\,\cancel{(n+1)!}\, x^{\cancel{n}+1}\,\cancel{(2n)}!}{\cancel{(2n+2)}!\,\cancel{n!}\,\cancel{n!}\,\cancel{x^n}} \right|$$

$$= \lim \left| \frac{(n+1)^2\, x}{(2n+1)(2n+2)} \right| = \lim \frac{(n+1)^2}{(2n+1)(2n+2)} |x| = \underline{\underline{\frac{1}{4}|x|}}$$

Konvergent når $|x| < 4 \iff \underline{x \in (-4,4)}.$

Endepunktene $x = \pm 4$ Svært vanskelig.

12.7. I det åpne konvergensintervallet kan potensrekken deriveres og integreres leddvis.

12.7.1. a) $f(x) = \sum_{n=0}^{\infty} n^2 x^n$ $\quad F(x) = \sum_{0}^{\infty} \frac{n^2}{n+1} x^{n+1}$

$$f'(x) = \sum_{n=1}^{\infty} n^2 \cdot n x^{n-1} = \sum_{n=1}^{\infty} n^3 x^{n-1} = \sum_{n=0}^{\infty} (n+1)^3 x^n$$

$$F(x) = \int_0^x f(t)\,dt = \int_0^x \left(\sum_0^{\infty} n^2 t^n\right) dt = \sum_0^{\infty} \int_0^x n^2 t^n\,dt$$

$$= \sum_0^{\infty} \frac{n^2}{n+1} t^{n+1} \Big|_0^x = \sum_0^{\infty} \frac{n^2}{n+1} x^{n+1} = \sum_1^{\infty} \frac{(n-1)^2}{n} x^n$$

12.7.2. Vi vet at den geometriske rekka

$$a + at + at^2 + \cdots$$

er konvergent for $|t|<1$ og har sum $\frac{a}{1-t}$.

a) $\sum_0^\infty x^{2n} = \underbrace{\frac{1}{1-x^2}}$ når $|x^2|<1 \iff |x|<1$.

$a = 1$
$t = x^2$

b) Integrerer hver side

Venstre side $\qquad \int \sum_0 x^{2n} = \underbrace{\sum_0^\infty \frac{x^{2n+1}}{2n+1}}$ for $|x|<1$.

Høyre side $\int_0^x \frac{1}{1-t^2}\,dt = \frac{1}{2}\int_0^x \frac{1}{1+t} + \frac{1}{1-t}\,dt$

$$= \frac{1}{2}\left(\ln|1+t| - \ln|1-t|\right) = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|\Bigg|_{t=0}^{t=x}$$

$$= \underbrace{\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|}$$

$$\ln\left|\frac{1+x}{1-x}\right| = 2\sum_0^\infty \frac{x^{2n+1}}{2n+1} \qquad |x|<1.$$

c) Setter inn $x = \frac{1}{2}$. V.S $\quad \ln\left|\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right| = \ln 3$

H.S. $\quad 2\sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)^{2n+1}}{2n+1} = \cancel{2}\sum_0^\infty \frac{1}{2^{2n+\cancel{1}}\cdot(2n+1)}$

$$\boxed{= \sum_0^\infty \frac{1}{2^{2n}(2n+1)} = \ln 3}$$

12.7.3.

a) $\sum_{n=0}^{\infty} x^{3n+2} = \sum_{n=0}^{\infty} \underbrace{x^2}_{a}(\underbrace{x^3}_{t})^n = \frac{x^2}{1-x^3}$

når $|x^3|<1 \iff |x|<1$

$|x|<1$.

b) Integrerer hver side

V.S $\sum_{0}^{\infty} \frac{x^{3n+3}}{3n+3} = \left(\sum_{1}^{\infty} \frac{x^{3n}}{3n}\right.$

H.S $\int_0^x \frac{\cancel{t^2}}{1-\cancel{t^3}} dt = \int_0^{x^3} \frac{\frac{1}{3}du}{1-u} = -\frac{1}{3}\ln|1-u|\Big|_0^{x^3} = -\frac{1}{3}\ln|1-x^3|$

$u = t^3$
$du = 3t^2 dt$

$= -\frac{1}{3}\ln(1-x^3)$

$\ln(1-x^3) = -\cancel{3}\sum_{1}^{\infty} \frac{x^{3n}}{\cancel{3}n} = -\sum_{1}^{\infty} \frac{x^{3n}}{n}$

c) Setter $x=-1$.

For $x=-1$ er rekka $\sum_{n=1}^{\infty} \frac{(-1)^{3n}}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

konvergent alternerende rekke. Abels teorem gir at

$\ln 2 = -\sum_{1}^{\infty} \frac{(-1)^n}{n} = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}$

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$

Taylor rekka til $f(x)$ i et punkt a

$$Tf(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Pb: · Hvor konvergere? Bruk FT

· Konvergere mot f? Må vise $R_n(x) \to 0$ i I.

$M = \max \{ f^{(n)}(t) \mid t$ mellom a og $x \}$.

$$|R_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}$$

12.8.1. b) Taylor rekka til $f(x)=\sin x$ i punktet $\frac{\pi}{4}$.

$$f\left(\frac{\pi}{4}\right)=\frac{1}{2}\sqrt{2}$$

$$f'(x)=\cos x\,,\quad f'\left(\frac{\pi}{4}\right)=\frac{1}{2}\sqrt{2}$$

$$f''(x)=-\sin x\,,\quad f''\left(\frac{\pi}{4}\right)=-\frac{1}{2}\sqrt{2}$$

$$f'''(x)=-\cos x\,,\quad f'''\left(\frac{\pi}{4}\right)=-\frac{1}{2}\sqrt{2}$$

$$f^{(4)}(x)=\sin x\,,\quad f^{(4)}\left(\frac{\pi}{2}\right)=\frac{1}{2}\sqrt{2}$$

$$Tf(x)=\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}\left(x-\frac{\pi}{4}\right)-\frac{1}{2}\cdot\frac{1}{2}\sqrt{2}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{6}\cdot\frac{1}{2}\sqrt{2}\left(x-\frac{\pi}{4}\right)^3\cdots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{\left[\frac{n}{2}\right]}\sqrt{2}}{2\cdot n!}\left(x-\frac{\pi}{4}\right)^n$$

$[a]$ = største hele tall $\leq a$

$$\lim\left|\frac{a_{n+1}}{a_n}\right|=\lim\left|\frac{\cancel{\sqrt{2}}\left(x-\frac{\pi}{4}\right)^{n+1}\cancel{2}\,\cancel{n!}}{\cancel{2}(n+1)\cancel{!}\,\cancel{\sqrt{2}}\,\cancel{\left(x-\frac{\pi}{4}\right)^n}}\right|=\lim\frac{1}{n+1}\left|x-\frac{\pi}{4}\right|$$

$$=0$$

Konvergens for alle x, dvs $I=(-\infty,\infty)$.

Gitt x, la $M=\max\left\{|f^{(n+1)}(t)|\ ;\ t \text{ mellom } \frac{\pi}{4} \text{ og } x\right\}\leq 1$

fordi $f^{(k)}(t)$ er $\pm\sin t$, $\pm\cos t$.

$$|R_n(x)|\leq\frac{1}{(n+1)!}\left|x-\frac{\pi}{4}\right|^{n+1}\longrightarrow 0 \text{ når } n\to\infty$$

$$\lim\frac{a_{n+1}}{a_n}=\lim\frac{\left|x-\frac{\pi}{4}\right|}{n+1}=0.\ \Rightarrow\ \lim a_n=0.$$

Dette viser at $Tf(x)$ konvergerer mot $f(x)=\sin x$.

c) Taylor rekka til $f(x) = x^3 - 2x^2 + 7x - 4$ i punktet -1.

$f(-1) = -1 - 2\cdot 1 - 7 - 4 = \underline{-14}$

$f'(x) = 3x^2 - 4x + 7$, $f'(-1) = 3 - 4(-1) + 7 = \underline{14}$

$f''(x) = 6x - 4$, $f''(-1) = -6 - 4 = \underline{-10}$.

$f'''(x) = 6$, $f'''(-1) = \underline{6}$

$f^{(4)}(x) = 0$

$\vdots$

$$Tf(x) = -14 + 14(x+1) - \frac{10}{2}(x+1)^2 + \frac{6}{6}(x+1)^3$$

$$= \underline{-14 + 14(x+1) - 5(x+1)^2 + (x+1)^3}$$

Konvergen for alle x, siden summen er endelig.

Når $n \geq 3$ er $R_n(x) = 0$ siden $f^{(n+1)}(x) \equiv 0$. (da $M = 0$)

så $\underline{Tf(x) = f(x)}$

12.8.3

c) $g(x) = \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots$

for alle x.

$$f(x) = \cos 3x^2 = \sum_{0}^{\infty} (-1)^n \frac{(3x^2)^{2n}}{(2n)!} =$$

$$\sum_{0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!} x^{4n}$$

12.8.15. $\sum_{0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}$.

a) Konvergensområdet?

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left| \frac{x^{2n+2} \cdot (2n+1)}{(2n+3)\, \cancel{x^{2n}}} \right| = \lim \frac{2n+1}{2n+3} |x|^2 = |x|^2.$$

Konvergent når $|x|^2 < 1 \Leftrightarrow |x| < 1$.

Endepunktene $x = \pm 1$: $\sum \frac{(-1)^n}{2n+1}$ konvergent alternerende rekke siden $a_n \to 0$ monotont.

Konvergensområdet : $[-1, 1]$.

b) Hva er summen $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}$?

$$\underbrace{\sum_0^{\infty} (-1)^n x^{2n} = \sum_0^{\infty} \underset{a=1}{1}\,(\underset{t}{-x^2})^n = \frac{1}{1+x^2}}$$ når

$|-x^2| = |x^2| < 1 \Leftrightarrow |x| < 1$.

Integrer begge sider

$$\boxed{\sum_0^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \int_0^x \frac{1}{1+t^2}\,dt = \arctan x}$$

$x \neq 0$, deler med x : $\displaystyle\sum_0^{\infty} (-1)^n \frac{x^{2n}}{2n+1} = \frac{\arctan x}{x}$

$x = 0$ er summen 1

$$\sum_0^{\infty} a_n x^n = a_0 + a_1 x + \dots$$

c) Finn $\arctan \frac{1}{2}$ med feil $< \frac{1}{100}$

$$\arctan \frac{1}{2} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{2n+1}}{(2n+1)} = \sum_0^{\infty} (-1)^n \frac{1}{2^{2n+1}(2n+1)} =$$

$$\frac{1}{2^1 \cdot 1} - \frac{1}{2^3 \cdot 3} + \frac{1}{2^5 \cdot 5} - \dots$$

$$= \underbrace{\frac{1}{2} - \frac{1}{24}} + \frac{1}{160}$$

Trenger bare to ledd, siden rekka er alternerende og feilen derfor mindre enn "neste ledd".

$$\arctan \frac{1}{2} \approx \frac{1}{2} - \frac{1}{24} = \frac{11}{24}.$$