6.4.4 Overflaten av en kule med radius R.

Vi bruker kulekoordinater

$$\bar{r}(\phi,\theta) = (R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi)$$

der $0 \le \phi \le \pi$ og $0 \le \theta \le 2\pi$

$$\frac{\partial \bar{r}}{\partial \phi} \times \frac{\partial \bar{r}}{\partial \theta} = \begin{vmatrix} i & j & k \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix}$$

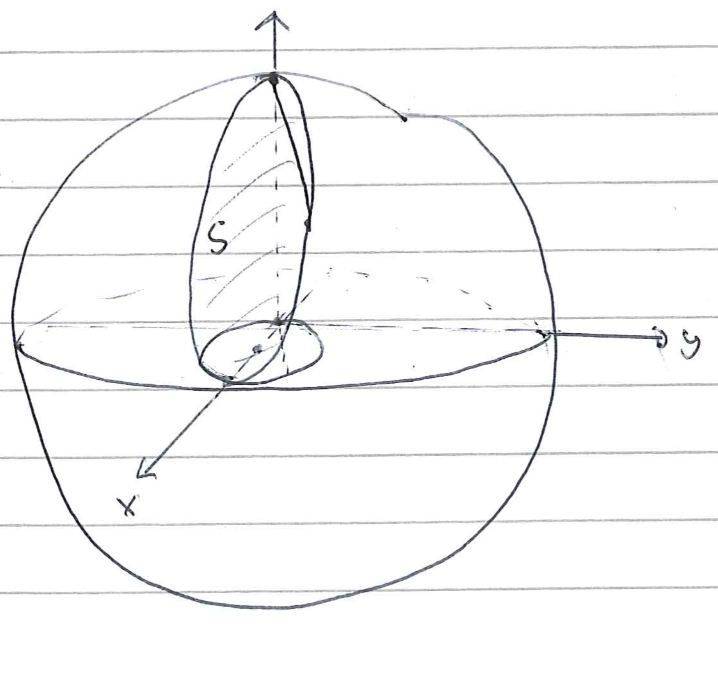
$$= (R^2\sin^2\phi\cos\theta,\ R^2\sin^2\phi\sin\theta,\ R^2\sin\phi\cos\phi)$$

$$\left|\frac{\partial \bar{r}}{\partial \phi} \times \frac{\partial \bar{r}}{\partial \theta}\right|^2 = R^4\sin^4\phi + R^4\sin^2\phi\cos^2\phi = R^4\sin^2\phi$$

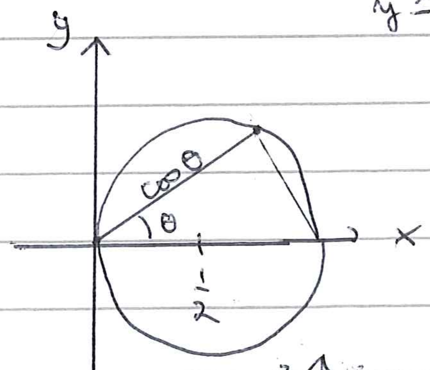
$$dS = R^2\sin\phi\, d\phi\, d\theta$$

$$A = \iint_S dS = \int_0^{2\pi}\left(\int_0^{\pi} R^2\sin\phi\, d\phi\right)d\theta = \int_0^{2\pi} 2R^2\, d\theta = \underline{\underline{4\pi R^2}}$$

6.4.7 Arealet av kuleflaten $x^2+y^2+z^2=1$ over disken $(x-\frac{1}{2})^2 + y^2 \le \frac{1}{4}$

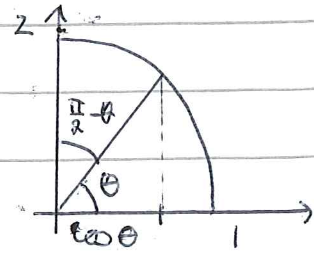


Vi ser på høyre halvdel, dvs. $y \ge 0$.



$0 \le \theta \le \frac{\pi}{2}$

$0 \le \phi \le \frac{\pi}{2} - \theta$

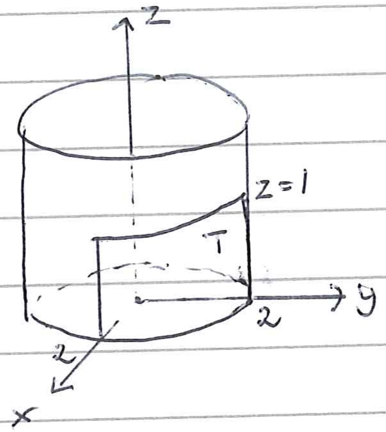


Dette gir

$$A = \iint_S dS = 2\int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}-\theta} \sin\phi \, d\phi \right) d\theta = 2\int_0^{\frac{\pi}{2}} (-\cos\phi) \Big|_0^{\frac{\pi}{2}-\theta} = 2\int_0^{\frac{\pi}{2}} (-\sin\theta + 1)\, d\theta$$

$$= 2\left(-1 + \frac{\pi}{2}\right) = \underline{\underline{\pi - 2}}.$$

6.4.10. $\iint_T xyz^2 \, dS$, T den delen av sylinderflaten $x^2+y^2=4$ der $x \geq 0$, $y \geq 0$ og $0 \leq z \leq 1$



Parameterisering $\bar{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$

$$\frac{\partial \bar{r}}{\partial \theta} \times \frac{\partial \bar{r}}{\partial z} = \begin{vmatrix} i & j & k \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos\theta, 2\sin\theta, 0)$$

$$dS = \left| \frac{\partial \bar{r}}{\partial \theta} \times \frac{\partial \bar{r}}{\partial z} \right| d\theta\, dz = 2\, d\theta\, dz$$

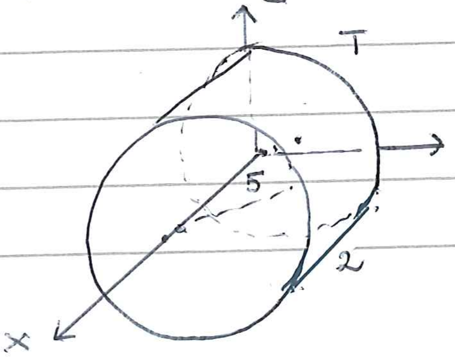
$$\iint_T xyz^2\, dS = \int_0^{\frac{\pi}{2}} \left(\int_0^1 2\cos\theta \cdot 2\sin\theta \, z^2 \, 2\,dz \right) d\theta$$

$$= \left(\int_0^{\frac{\pi}{2}} 8\sin\theta\cos\theta \, d\theta \right) \left(\int_0^1 z^2 dz \right) = (4\sin^2\theta)\Big|_0^{\frac{\pi}{2}} \cdot \frac{1}{3} = \underline{\underline{\frac{4}{3}}}$$

6.4.12 T sylinderflaten parametrisert ved

$\bar{r}(u,v) = (u, 5\cos v, 5\sin v)$ $\quad u \in [0,2] \quad v \in [0, 2\pi]$

Sylinder i x-retning med radius 5, lengde 2



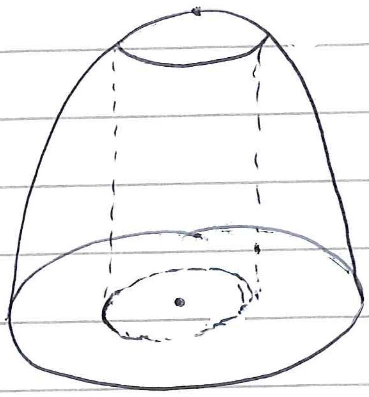
Som i forrige oppgave får vi

$dS = 5\, du\, dv$

$$\iint_T x\, dS = \int_0^{2\pi} \left(\int_0^2 u \cdot 5\, du \right) dv = \int_0^{2\pi} \frac{5}{2} u^2 \Big|_0^2 dv$$

$$= \int_0^{2\pi} 10\, dv = 10 \cdot 2\pi = \underline{\underline{20\pi}}$$

6.4.17 D ligger over xy-planet (dvs. $z \geq 0$), inne i paraboloiden $z = 4 - x^2 - y^2$ og sylinderen $x^2 + y^2 = 1$



a) Volumet til D

$$V = \iint_{D(0,1)} z\,dx\,dy = \int_0^1 \left(\int_0^{2\pi} (4 - r^2) r \, d\theta \right) dr$$

$$= 2\pi \int_0^1 4r - r^3 \, dr = 2\pi \left(4 \cdot \frac{1}{2} - \frac{1}{4} \right) = \underline{\underline{\frac{7}{2}\pi}}$$

b) Toppflaten er parametrisert ved

$$\bar{r}(r, \theta) = (r\cos\theta, r\sin\theta, 4 - r^2)$$

$$\frac{\partial \bar{r}}{\partial r} \times \frac{\partial \bar{r}}{\partial \theta} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2\cos\theta, 2r^2\sin\theta, r)$$

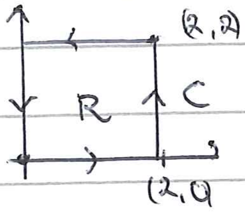
$$dS = \left| \frac{\partial \bar{r}}{\partial r} \times \frac{\partial \bar{r}}{\partial \theta} \right| dr\,d\theta = \sqrt{4r^4 + r^2}\, dr\,d\theta = r\sqrt{4r^2 + 1}\, dr\,d\theta$$

$$A = \int_0^1 \left(\int_0^{2\pi} r\sqrt{4r^2+1}\, d\theta \right) dr = 2\pi \int_0^1 r(4r^2+1)^{\frac{1}{2}} dr$$

$$= 2\pi \left((4r^2+1)^{3/2} \cdot \frac{2}{3} \cdot \frac{1}{8} \right) \Big|_0^1 = \frac{\pi}{6} (4r^2+1)^{3/2} \Big|_0^1 = \frac{\pi}{6} \left(5^{3/2} - 1^{3/2} \right) = \underline{\underline{\frac{\pi}{6}(5\sqrt{5} - 1)}}$$

6.5.1 Bruk Green $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

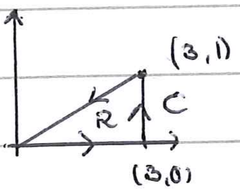
a) $\int_C (x^2+y)\,dx + x^2 y\,dy$



$= \iint_R 2xy - 1\,dx\,dy$

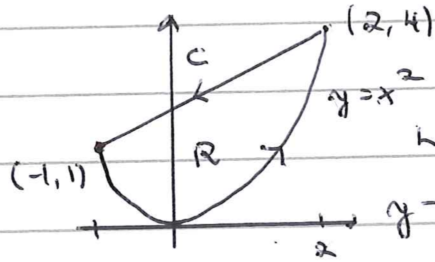
$= \int_0^2 \left(\int_0^2 2xy - 1\,dy \right) dx = \int_0^2 xy^2 - y \Big|_{y=0}^{2} dx = \int_0^2 4x - 2\,dx = 2x^2 - 2x \Big|_0^2 = 8 - 4 = \underline{\underline{4}}$

b) $\int_C x^2 y^3\,dx + x^3 y^2\,dy$



$= \iint_R (3x^2y^2 - 3x^2y^2)\,dx\,dy = \iint_R 0\,dx\,dy = \underline{\underline{0}}$

d) $\int_C (x^2 y + x e^x)\,dx + (x y^3 + e^{\sin y})\,dy$



Linja!

$y - 1 = \frac{3}{3}(x+1)$

$y = x + 2$

$= \iint_R (y^3 - x^2)\,dx\,dy$

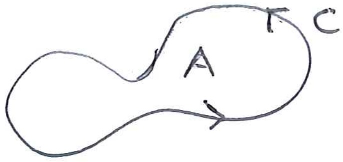
$= \int_{-1}^{2} \left(\int_{x^2}^{x+2} y^3 - x^2\,dy \right) dx = \int_{-1}^{2} \frac{1}{4}y^4 - x^2 y \Big|_{y=x^2}^{y=x+2} dx = \int_{-1}^{2} \frac{1}{4}(x+2)^4 - x^2(x+2) - \frac{1}{4}x^8 + x^4\,dx$

$= \int_{-1}^{2} \frac{1}{4}(x+2)^4 - x^3 - 2x^2 - \frac{1}{4}x^8 + x^4\,dx = \frac{1}{20}(x+2)^5 - \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{36}x^9 + \frac{1}{5}x^5 \Big|_{-1}^{2}$

$= \frac{4^5}{20} - 4 - \frac{16}{3} - \frac{512}{36} + \frac{32}{5} - \left(\frac{1}{20} - \frac{1}{4} + \frac{2}{3} + \frac{1}{36} - \frac{1}{5} \right)$

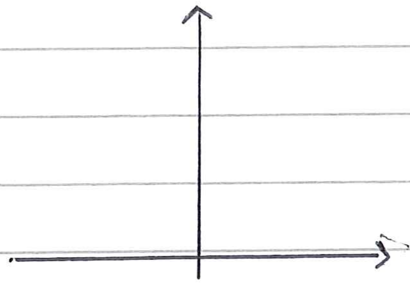
$= \frac{1024 + 128 - 1 + 4}{20} - 4 + \frac{-16 - 2}{3} + \frac{-512 + 9 - 1}{36} = \frac{1155}{20} - 4 - 6 - \frac{504}{36}$

$= \frac{231}{4} - 10 - 14 = \frac{231}{4} - 24 = \frac{231 - 96}{4} = \underline{\underline{\frac{135}{4}}}$



$$A=\iint dx\,dy = \int_C x\,dy = -\int_C y\,dx$$

6.5.2



Skisse MATLAB. se neste side

$\vec{r}(t)=(t\sin t,\ 2\pi t-t^2)$

$t\in[0,2\pi]$

$$A=\int_C x\,dy = \int_0^{2\pi} t\sin t\,(2\pi-2t)\,dt = \int_0^{2\pi}\underbrace{(2\pi t-2t^2)}_{u}\,\underbrace{\sin t}_{v'}\,dt$$

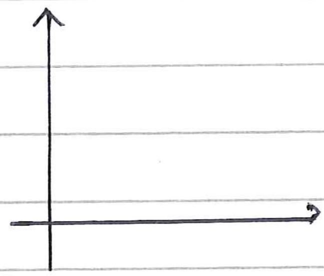
$u'=2\pi-4t \qquad v=-\cos t$

$$=(2\pi t-2t^2)(-\cos t)\Big|_0^{2\pi} + \int_0^{2\pi}\underbrace{(2\pi-4t)}_{u}\,\underbrace{\cos t}_{v'}\,dt$$

$u'=-4 \qquad v=\sin t$

$$=(4\pi^2-8\pi^2)(-1) + (2\pi-4t)\sin t\Big|_0^{2\pi} + \int_0^{2\pi}4\sin t\,dt = 4\pi^2+0+0 = \underline{\underline{4\pi^2}}$$

6.5.3



Skisse MATLAB se 2 sider bak.

$\vec{r}(t)=(\sin 2t,\ t\cos t)$, $t\in[0,\frac{\pi}{2}]$

$$A=-\int_C y\,dx = \int_0^{\frac{\pi}{2}} -t\cos t\cdot(2\cos 2t)\,dt$$

$$=\int_0^{\frac{\pi}{2}}\underbrace{-2t}_{u}\,\underbrace{\cos t(1-2\sin^2 t)}_{v'}\,dt = -2t\left(\sin t-\tfrac{2}{3}\sin^3 t\right)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 2\sin t-\tfrac{4}{3}\sin^3 t$$

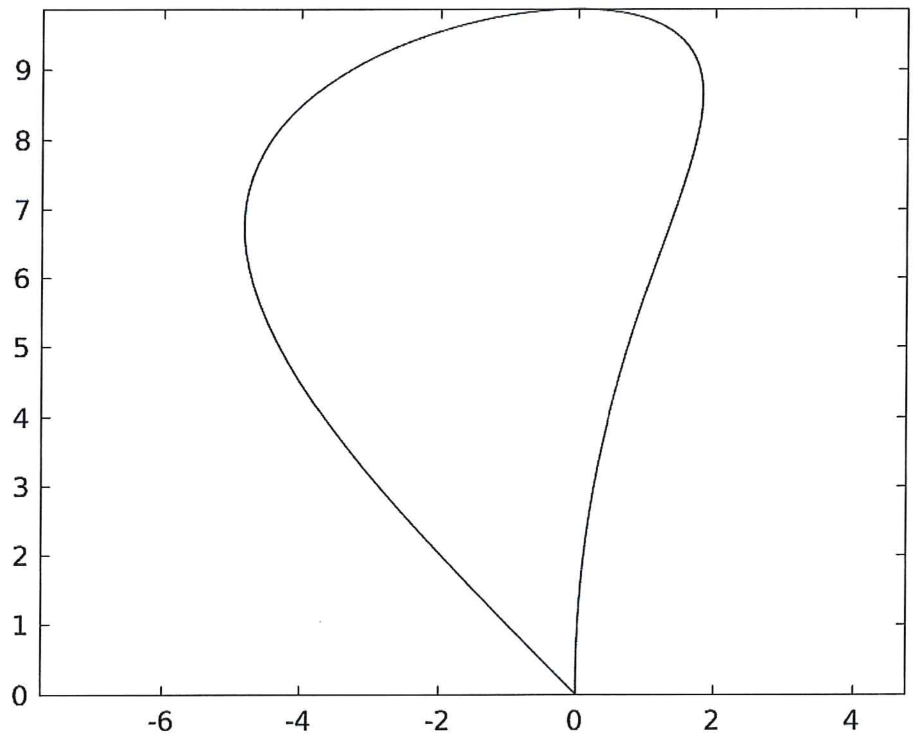
$u'=-2 \qquad v=\sin t-\frac{2}{3}\sin^3 t$

$$=-\pi\left(1-\tfrac{2}{3}\right)+2-\tfrac{4}{3}\int_0^{\frac{\pi}{2}}\sin t(1-\cos^2 t)\,dt = -\tfrac{\pi}{3}+2-\tfrac{4}{3}\left(-\cos t+\tfrac{1}{3}\cos^3 t\right)\Big|_0^{\frac{\pi}{2}}$$

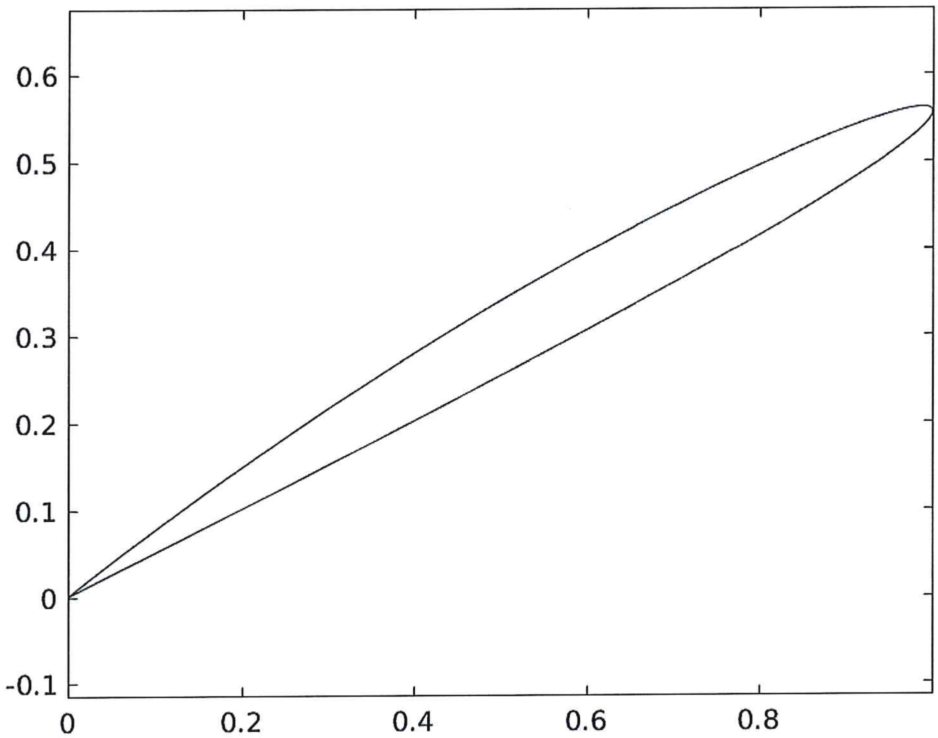
$$=-\tfrac{\pi}{3}+2-\tfrac{4}{3}\left(0+0-\left(-1+\tfrac{1}{3}\right)\right) = -\tfrac{\pi}{3}+2-\tfrac{4}{3}\cdot\left(\tfrac{2}{3}\right) = -\tfrac{\pi}{3}+2-\tfrac{8}{9} = \underline{\underline{\tfrac{10}{9}-\tfrac{\pi}{3}}}$$

Oppgave 6.5.2.

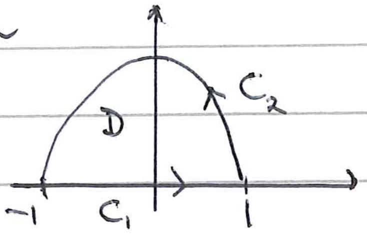
```
t = linspace(0,2*pi,100);
x = t.*sin(t);
y = t.*(2*pi-t);
plot(x,y)
axis('equal')
```



Oppgave 6.5.3



6.5.8 D avgrenset av $y = 1 - x^2$ og x-aksen



a) $\int_C -y\,dx + x^2\,dy$, direkte utregning

C_1 : $\bar{r}_1(t) = (t, 0)$, $-1 \le t \le 1$

$$\int_{C_1} -y\,dx + x^2\,dy = \int_{-1}^{1} 0 + t^2 \cdot 0 \, dt = 0$$

C_2 : $\bar{r}_2(t) = (t, 1 - t^2)$ $-1 \le t \le 1$ orientert motsatt vei

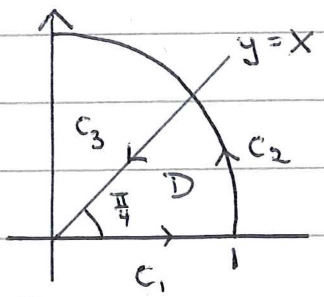
$$\int_{C_2} -y\,dx + x^2\,dy = -\int_{-1}^{1} (t^2 - 1) \cdot 1 + t^2(-2t)\,dt = \int_{-1}^{1} 1 - t^2 + 2t^3\,dt$$

$$= \left(t - \frac{1}{3}t^3 + \frac{1}{2}t^4\right)\Big|_{-1}^{1} = \left(1 - \frac{1}{3} + \frac{1}{2}\right) - \left(-1 + \frac{1}{3} + \frac{1}{2}\right) = \underline{\underline{\frac{4}{3}}}$$

b) Green gir

$$\int_C -y\,dx + x^2\,dy = \iint_D (2x + 1)\,dx\,dy \overset{\text{Pga symmetri}}{=} \iint_D dx\,dy = \int_{-1}^{1} 1 - x^2\,dx = 2 - \frac{2}{3} = \underline{\underline{\frac{4}{3}}}$$

6.5.10



$$I = \iint_D (x + y^2)\,dx\,dy = \int_0^{\frac{\pi}{4}} \int_0^1 (r\cos\theta + r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\int_0^1 r^2\cos\theta + r^3\sin^2\theta\,dr \right) d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{3}\cos\theta + \frac{1}{4}\sin^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{3}\cos\theta + \frac{1}{4} \cdot \frac{1}{2}(1 - \cos 2\theta)\,d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{3}\cos\theta + \frac{1}{8} - \frac{1}{8}\cos 2\theta\,d\theta$$

$$= \frac{1}{3}\sin\theta + \frac{1}{8}\theta - \frac{1}{16}\sin 2\theta \Big|_0^{\frac{\pi}{4}} = \frac{1}{3}\cdot\frac{1}{2}\sqrt{2} + \frac{1}{8}\cdot\frac{\pi}{4} - \frac{1}{16} = \underline{\underline{\frac{1}{6}\sqrt{2} + \frac{\pi}{32} - \frac{1}{16}}}$$

b) Vi må ha $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x + y^2$. Kan f. eks. velge $Q = \frac{1}{2}x^2 + xy^2$, $P = 0$

Da er $I = \int_C Q\,dy = \int_C (\frac{1}{2}x^2 + xy^2)\,dy$

C_1: $\bar{r}_1(t) = (t, 0) \Rightarrow dy = 0\,dt$, så $\int_{C_1} Q\,dy = 0$

C_2: $\bar{r}_2(\theta) = (\cos\theta, \sin\theta)$, $0 \le \theta \le \frac{\pi}{4}$

$$\int_{C_2} (\tfrac{1}{2}x^2 + xy^2)\,dy = \int_0^{\frac{\pi}{4}} (\tfrac{1}{2}\cos^2\theta + \cos\theta \cdot \sin^2\theta)\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\tfrac{1}{2} - \tfrac{1}{2}\sin^2\theta)\cos\theta + \tfrac{1}{4}\sin^2 2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{1}{2}\cos\theta - \tfrac{1}{2}\sin^2\theta\cos\theta + \tfrac{1}{4}(\tfrac{1}{2} - \tfrac{1}{2}\cos 4\theta)\,d\theta$$

$$= \tfrac{1}{2}\sin\theta - \tfrac{1}{6}\sin^3\theta + \tfrac{1}{8}\theta - \tfrac{1}{32}\sin 4\theta \Big|_0^{\frac{\pi}{4}} = \tfrac{1}{2}\cdot\tfrac{1}{2}\sqrt{2} - \tfrac{1}{6}\cdot\tfrac{1}{4}\sqrt{2} + \tfrac{\pi}{32}$$

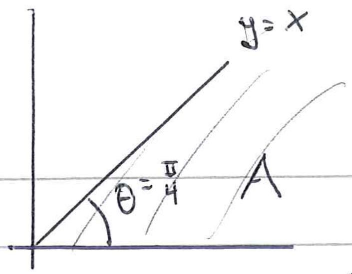
$$= (\tfrac{1}{4} - \tfrac{1}{24})\sqrt{2} + \tfrac{\pi}{32} = \tfrac{5}{24}\sqrt{2} + \tfrac{\pi}{32}$$

C_3: $\bar{r}(t) = (t, t)$ $\quad 0 \le t \le \frac{1}{2}\sqrt{2}$ orientert motsatt vei

$$\int_{C_3} (\tfrac{1}{2}x^2 + xy^2)\,dy = -\int_0^{\frac{1}{2}\sqrt{2}} (\tfrac{1}{2}t^2 + t^3)\,dt = -\tfrac{1}{6}t^3 - \tfrac{1}{4}t^4 \Big|_0^{\frac{1}{2}\sqrt{2}} = -\tfrac{1}{6}\cdot\tfrac{1}{4}\sqrt{2} - \tfrac{1}{4}\cdot\tfrac{1}{4}$$

$$= -\tfrac{1}{24}\sqrt{2} - \tfrac{1}{16}$$

$$I = 0 + \tfrac{5}{24}\sqrt{2} + \tfrac{\pi}{32} - \tfrac{1}{24}\sqrt{2} - \tfrac{1}{16} = \underline{\underline{\tfrac{1}{6}\sqrt{2} + \tfrac{\pi}{32} - \tfrac{1}{16}}}$$



6.8.1

$$\iint_A e^{-(x^2+y^2)}\,dx\,dy = \lim_{n\to\infty}\iint_{A\cap B(0,n)} e^{-(x^2+y^2)}\,dx\,dy = \lim_{n\to\infty}\int_0^n\left(\int_0^{\frac{\pi}{4}} e^{-r^2} r\,d\theta\right)dr$$

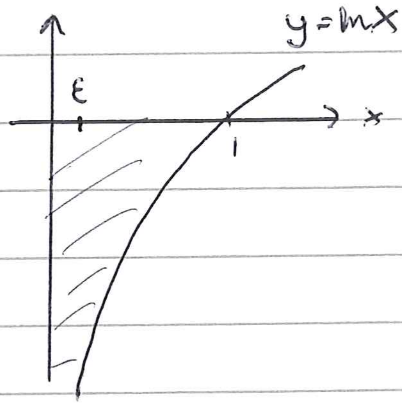
$$= \lim_{n\to\infty}\int_0^n \frac{\pi}{4} r e^{-r^2}\,dr = \lim_{n\to\infty}\frac{\pi}{4}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^n = \lim_{n\to\infty}\frac{\pi}{4}\left(-\frac{1}{2}e^{-n^2}+\frac{1}{2}\right) = \underline{\underline{\frac{\pi}{8}}}$$

6.8.2

$$\iint_{\mathbb{R}^2}\frac{1}{1+x^2+y^2}\,dx\,dy = \lim_{n\to\infty}\int_0^n\left(\int_0^{2\pi}\frac{r\,d\theta}{1+r^2}\right)dr$$

$$= \lim_{n\to\infty}\int_0^n 2\pi\frac{r}{1+r^2}\,dr = \lim_{n\to\infty} 2\pi\left(\frac{1}{2}\ln(1+r^2)\right)\Big|_0^n = \lim_{n\to\infty}\pi\ln(1+n^2) = \infty.$$

Integralet divergerer.



6.8.3 A området i fjerde kvadrant mellom y-aksen og $y = \ln x$.
Vi ser på integralet over A_ε, den delen av A der $x \geq \varepsilon$ og lar $\varepsilon \to 0$.

$$\iint_A x\,dx\,dy = \lim_{\varepsilon\to 0}\int_\varepsilon^1\left(\int_{\ln x}^0 x\,dy\right)dx$$

$$= \lim_{\varepsilon\to 0}\int_\varepsilon^1 (-x\ln x)\,dx = \lim_{\varepsilon\to 0}\left(-\frac{1}{2}x^2\ln x + \frac{1}{4}x^2\right)\Big|_\varepsilon^1$$

Delvis integrasjon

$$= \lim_{\varepsilon\to 0}\frac{1}{4} - \left(-\frac{1}{2}\varepsilon^2\ln\varepsilon + \frac{1}{4}\varepsilon^2\right) = \underline{\underline{\frac{1}{4}}}$$

Konvergent.