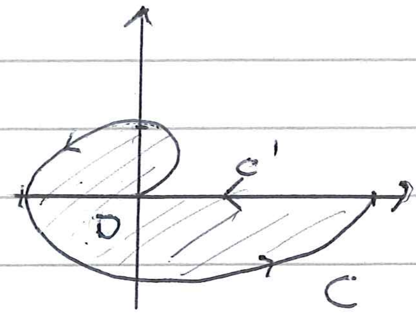
14/6 - 2017 - Oppgave 1.

$\bar{r}(t) = (t\cos t, t\sin t) \qquad t \in [0, 2\pi]$

$\bar{F}(x,y) = (-y, x)$



a) $\bar{r}'(t) = (\cos t - t\sin t, \sin t + t\cos t)$

$\bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) = (-t\sin t, t\cos t) \cdot (\cos t - t\sin t, \sin t + t\cos t)$

$= -t\sin t\cos t + t^2\sin^2 t + t\cos t\sin t + t^2\cos^2 t$

$= t^2(\sin^2 t + \cos^2 t) = t^2$

$$\int_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} t^2\,dt = \frac{1}{3}t^3\Big|_0^{2\pi} = \underline{\underline{\frac{8}{3}\pi^3}}$$

b) To måter:

1) Polarkoordinater:

$$A(D) = \iint_D dx\,dy = \int_0^{2\pi}\left(\int_0^{|r(t)|} r\,dr\right)dt = \int_0^{2\pi}\left(\int_0^{t} r\,dr\right)dt = \int_0^{2\pi}\frac{1}{2}t^2\,dt =$$

$$\frac{1}{6}t^3\Big|_0^{2\pi} = \underline{\underline{\frac{4}{3}\pi^3}}$$

2) Greens teorem: C' er parametrisert ved $(1-t, 0)$ så $-y\,dx + x\,dy = -0\cdot(-dt) + (1-t)\cdot 0\,dt = 0\,dt$. Altså gir Green og a)

$$\frac{8}{3}\pi^3 = \int_C -y\,dx + x\,dy = \int_{C\cup C'} -y\,dx + x\,dy \overset{\text{Green}}{=} \iint_D (1+1)\,dx\,dy = 2A(D)$$

altså $A(D) = \underline{\underline{\frac{4}{3}\pi^3}}$

18/8 - 2017 - Oppgave 1.

$$A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

a) Egenverdier

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \lambda - \frac{1}{3} \end{vmatrix} = (\lambda - \tfrac{1}{3})^2 - (\tfrac{2}{3})^2 = 0$$

$$\lambda - \frac{1}{3} = \pm \frac{2}{3} \Rightarrow \lambda = \frac{1}{3} \pm \frac{2}{3} = \begin{cases} 1 \\ -\frac{1}{3} \end{cases}$$

Egenvektoren $\lambda_1 = 1$:

$\frac{2}{3}x - \frac{2}{3}y = 0$. y velges fritt $y = s$. Som gir $x = s$

$x = y$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad s \in \mathbb{R}$$

$\lambda_2 = -\frac{1}{3}$:

$-\frac{2}{3}x - \frac{2}{3}y = 0$ y velges fritt $y = t$. Får $x = -t$

$x = -y$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad t \in \mathbb{R}$$

b) Skriv $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ som lineær kombinasjon av egenvektorer:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} s - t = 1 \\ s + t = 0 \end{matrix} \qquad s = \frac{1}{2},\ t = -\frac{1}{2}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = A^n \left(\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$

$$= \frac{1}{2} \cdot 1^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \cdot \left(-\frac{1}{3}\right)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow[n\to\infty]{} \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}}}$$

18/8 - 2017 - Oppgave 4

$$\bar{F} = \left(\frac{y}{x^2+y^2+1}, -\frac{x}{x^2+y^2+1} \right) =: (F_1, F_2)$$

a)

$$\frac{\partial F_1}{\partial x} = \frac{-2xy}{(x^2+y^2+1)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{1 \cdot (x^2+y^2+1) - y(2y)}{(x^2+y^2+1)^2} = \frac{x^2-y^2+1}{(x^2+y^2+1)^2}$$

$$\left.\begin{aligned} \frac{\partial F_2}{\partial x} &= \frac{x^2-y^2-1}{(x^2+y^2+1)^2} \\ \frac{\partial F_2}{\partial y} &= \frac{2xy}{(x^2+y^2+1)^2} \end{aligned}\right\} \text{av symmetrigrunner}$$

$$\bar{F}'(x,y) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \frac{1}{(x^2+y^2+1)^2} \begin{pmatrix} -2xy & x^2-y^2+1 \\ x^2-y^2-1 & 2xy \end{pmatrix}$$

$$\bar{F}(0,0) = (0,0) \qquad \bar{F}'(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Linearisering $\bar{F}(0,0) + \bar{F}'(0,0)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{\underline{\begin{pmatrix} y \\ -x \end{pmatrix}}}$

(eller $(y, -x)$ om vi vil)

b) $\bar{F}$ er ikke konservativ siden $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$

c) $\bar{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ Enhetssirkelen

$$\bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) = \left(\frac{\sin t}{2}, -\frac{\cos t}{2}\right) \cdot (-\sin t, \cos t)$$

$$= -\frac{1}{2}\sin^2 t - \frac{1}{2}\cos^2 t = \underline{-\frac{1}{2}}$$

$$\int_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} -\frac{1}{2}\, dt = -\frac{1}{2} \cdot 2\pi = \underline{\underline{-\pi}}$$

(Dette viser også at $\bar{F}$ ikke er konservativ, siden C er lukket og $\int_C \bar{F} \cdot d\bar{r} \neq 0$)

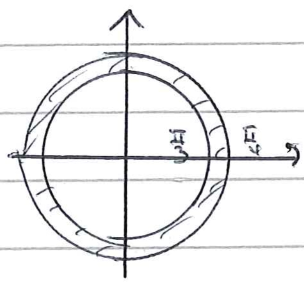
Kan også bruke Green:

$$\int_C F_1\, dx + F_2\, dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \iint_D \frac{-2}{(x^2+y^2+1)^2}\, dx\, dy$$

$$\overset{\text{polar}}{=} \int_0^{2\pi} \left(\int_0^1 \frac{-2r}{(r^2+1)^2}\, dr \right) d\theta = \int_0^{2\pi} \left. \frac{1}{r^2+1} \right|_0^1 d\theta = \int_0^{2\pi} \left(\frac{1}{2} - 1\right) d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\, d\theta = -\frac{1}{2} \cdot 2\pi = \underline{\underline{-\pi}}$$

18/8-2017 - Oppgave 5 $A = \left\{ (x, y);\ \frac{\pi}{4} \le \sqrt{x^2+y^2} \le \frac{\pi}{3} \right\}$



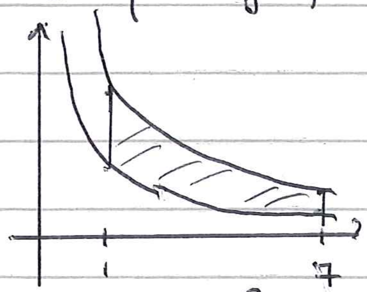
Annulus med indre radius $\frac{\pi}{4}$, ytre $\frac{\pi}{3}$.
Bruker polarkoordinater, $\frac{\pi}{4} \le r \le \frac{\pi}{3}$, $0 \le \theta \le 2\pi$

$$\iint_A \frac{\tan\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\, dx\, dy = \int_{\pi/4}^{\pi/3} \left(\int_0^{2\pi} \frac{\tan r}{r}\, r\, d\theta \right) dr = 2\pi \int_{\pi/4}^{\pi/3} \tan r\, dr$$

$$2\pi\,(-\ln \cos x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = 2\pi\left(\ln \tfrac{1}{2}\sqrt{2} - \ln \tfrac{1}{2}\right) = 2\pi \ln\sqrt{2} = 2\pi \ln 2^{\frac{1}{2}}$$

$$= \underline{\underline{\pi \ln 2}}$$

b) $B = \left\{ (x, y) ;\ 1 \le x \le 7 ,\ \frac{1}{x} \le y \le \frac{2}{x} \right\}$



$$\iint_B x y^2 \, dx\, dy = \int_1^7 \left(\int_{\frac{1}{x}}^{\frac{2}{x}} x y^2 \, dy \right) dx$$

$$= \int_1^7 \frac{1}{3} x y^3 \Big|_{y=\frac{1}{x}}^{\frac{2}{x}} dx = \int_1^7 \frac{1}{3} x \left(\frac{8}{x^3} - \frac{1}{x^3} \right) dx = \int_1^7 \frac{7}{3} x^{-2} \, dx$$

$$= -\frac{7}{3} x^{-1} \Big|_1^7 = \frac{7}{3}\left(1 - \frac{1}{7}\right) = \frac{7}{3} \cdot \frac{6}{7} = \underline{\underline{2.}}$$