6.9.1

a) $\iiint\limits_A xyz \, dx\, dy\, dz$ $\quad A = [0,1]\times[0,1]\times[0,1]$

$= \int_0^1 \left(\int_0^1 \left(\int_0^1 xyz \right) dx\, dy\, dz = \left(\int_0^1 x\,dx\right)\left(\int_0^1 y\,dy\right)\left(\int_0^1 z\,dz\right) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$

$\uparrow$

Når vi har faste grenser og en funksjon som er et produkt av funksjoner i hver variabel, er dette lov.

b) $\iiint\limits_A x + y e^z \, dx\, dy\, dz$ $\quad A = [-1,1]\times[0,1]\times[1,2]$

$= \left(\int_{-1}^1 x\,dx\right)\int_0^1 dy \cdot \int_1^2 dz + \int_{-1}^1 dx \cdot \int_0^1 y\,dy \cdot \int_1^2 e^z dz = 0\cdot 1\cdot 1 + 2\cdot\frac{1}{2}\cdot(e^2 - e)$

$= \underline{e^2 - e}$

c) $\iiint\limits_A zy \cos(xy)\, dx\, dy\, dz$ $\quad A = [1,2]\times[\pi, 2\pi]\times[0,1]$

$= \int_0^1 \left(\int_\pi^{2\pi} \left(\int_1^2 zy\cos(xy)\,dx \right) dy\right) dz = \int_0^1 \left(\int_\pi^{2\pi} \left[z\sin(xy) \right]_{x=1}^{x=2} dy \right) dz$

$= \int_0^1\int_\pi^{2\pi} z(\sin 2y - \sin y)\, dy\, dz = \left(\int_0^1 z\,dz\right)\left(\int_\pi^{2\pi} \sin 2y - \sin y \, dy\right)$

$= \frac{1}{2}\cdot\left[-\frac{1}{2}\cos 2y + \cos y\right]_\pi^{2\pi} = \frac{1}{2}\left[-\frac{1}{2}\cos 4\pi + \cos 2\pi + \frac{1}{2}\cos 2\pi - \cos\pi\right]$

$= \frac{1}{2}\left[-\frac{1}{2} + 1 + \frac{1}{2} + 1\right] = 1.$

d) $\iiint_A (x+y+z)\,dx\,dy\,dz$ $\quad A=[0,1]\times[0,2]\times[0,3]$

$$\iiint_A x\,dx\,dy\,dz = \left(\int_0^1 x\,dx\right)\left(\int_0^2 dy\right)\left(\int_0^3 dz\right) = \tfrac12\cdot 2\cdot 3 = 3$$

$$\iiint y = \left(\int_0^1 dx\right)\left(\int_0^2 y\,dy\right)\int_0^3 dz = 1\cdot\tfrac12 2^2\cdot 3 = 6$$

$$\iiint z = \left(\int dx\right)\left(\int dy\right)\int_0^3 z\,dz = 1\cdot 2\cdot\tfrac12 3^2 = 9.$$

$$\iiint x+y+z = \underline{18.}$$

eller

$$\int_0^3\left(\int_0^2\left(\int_0^1 (x+y+z)\,dx\right)dy\right)dz = \int_0^3\int_0^2\left[\tfrac12 x^2+xy+xz\right]_0^1 dy\,dz =$$

$$\int_0^3\left(\int_0^2\left(\tfrac12+y+z\right)dy\right)dz = \int_0^3 \left(\tfrac12 y+\tfrac12 y^2+yz\right)\Big|_0^2 dz =$$

$$\int_0^3 1+2+2z\,dz = 3z+z^2\Big|_0^3 = 9+9 = \underline{18.}$$

e) $\iiint_A (\sqrt{y}-3z)\,dx\,dy\,dz$ $\quad A=[2,3]\times[0,1]\times[-1,1]$

$$= \int_{-1}^1\left(\int_0^1\left(\int_2^3 \sqrt{y}-3z\,dx\right)dy\right)dz = \int_{-1}^1\left(\int_0^1 \sqrt{y}-3z\,dy\right)dz$$

$$= \int_{-1}^1\left[\tfrac23 y^{3/2}-3yz\right]_0^1 dz = \int_{-1}^1 \tfrac23-3z\,dz = \tfrac23\cdot 2 = \underline{\underline{\tfrac43}}$$

Ø. 9.2.

a) $\iiint_A xy+z\,dx\,dy\,dz$ $\quad A=\{0\le x\le 1,\ 0\le y\le 2,\ 0\le z\le x^2y\}$

$$= \int_0^1\left(\int_0^2\left(\int_0^{x^2y} xy+z\,dz\right)dy\right)dx = \int_0^1\left(\int_0^2 xyz+\tfrac12 z^2\Big|_0^{x^2y} dy\right)dx = \int_0^1\left(\int_0^2 x^3y^2+\tfrac12 x^4y^2\,dy\right)dz$$

$$= \int_0^1 \tfrac13 x^3y^3+\tfrac16 x^4y^3\Big|_0^2 dx = \int_0^1 \tfrac83 x^3+\tfrac86 x^4\,dx = \tfrac83\cdot\tfrac14+\tfrac43\cdot\tfrac15 = \tfrac23+\tfrac4{15} = \tfrac{14}{15}$$

b) $\iiint_A z\,dx\,dy\,dz$ $A=\{(x,y,z);\ 0\le x\le 2,\ 0\le y\le \sqrt{x},\ -y^2\le z\le xy\}$

$$= \int_0^2 \left(\int_0^{\sqrt{x}} \left(\int_{-y^2}^{xy} z\,dz \right) dy \right) dx = \int_0^2 \left(\int_0^{\sqrt{x}} \left[\frac12 z^2\right]_{-y^2}^{xy} dy \right) dx = \frac12 \int_0^2 \left(\int_0^{\sqrt{x}} x^2y^2 - y^4\,dy \right) dx$$

$$= \frac12 \int_0^2 \left[\frac13 x^2 y^3 - \frac15 y^5\right]_0^{\sqrt{x}} dx = \frac12 \int_0^2 \frac13 x^{\frac72} - \frac15 x^{5/2}\,dx =$$

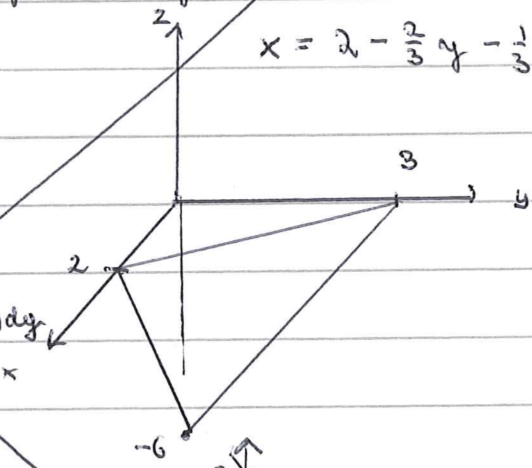
$$\frac12 \left(\frac13 \cdot \frac29 x^{\frac92} - \frac15 \cdot \frac27 x^{7/2} \right) \Big|_0^2 = \frac{1}{27} \cdot 2^{9/2} - \frac{1}{35} 2^{7/2} = \frac{16}{27}\sqrt{2} - \frac{8}{35}\sqrt{2}$$

$$= \frac{35\cdot 16 - 27\cdot 8}{27\cdot 35}\sqrt{2} = \frac{344}{945}\sqrt{2}$$

d) $\iiint_A 3y^2 - 3z\,dx\,dy\,dz$ A = området avgrenset av koordinatplan og $3x + 2y - z = 6$.

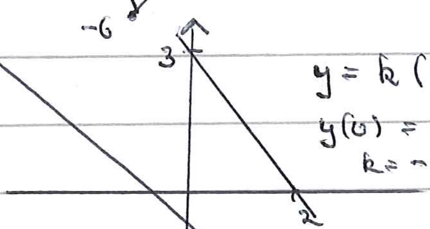
$x = 2 - \frac23 y - \frac13 z$

$$\int_0^3 \left(\int_0^{2y-6} \left(\int_0^{2-\frac23 y - \frac13 z} 3y^2 - 3z\,dx \right) dz \right) dy$$

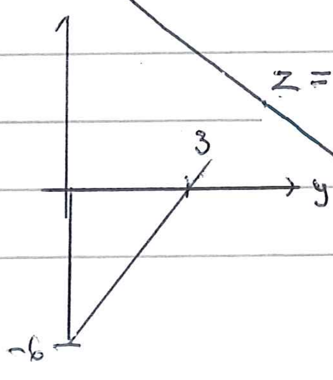


$$= \int_0^3 \left(\int_0^{2y-6} 6y^2 - 2y^3 - y^2 z - 6z + 2yz + z^2\,dz \right) dy$$

$$= \int_0^3 (2y-6)(6y^2-2y^3) + \frac12 (2y-6)^2 (2y-6-y^2)$$



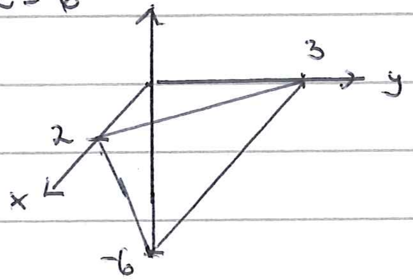
$y = k(x-2)$
$y(0) = -2k = 3$
$k = -\frac32,\ y = 3 - \frac32 x$



$z = 2y - 6$

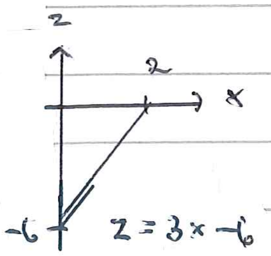
d) $\iiint_A 3y^2 - 3z \; dx\,dy\,dz$

A = området avgrenset av koordinatplana og $3x + 2y - z = 6$



Deler opp

$$\iiint_A 3y^2 \; dx\,dy\,dz$$



$z = 3x - 6$

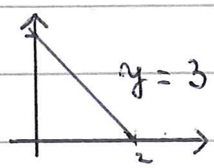
$y = 3 - \frac{3}{2}x + \frac{1}{2}z$

$$= \int_0^2 \Big(\int_{3x-6}^{0} \Big(\int_0^{3-\frac{3}{2}x+\frac{1}{2}z} 3y^2 \, dy \Big) dz \Big) dx$$

$$= \int_0^2 \Big(\int_{3x-6}^{0} \big(3 - \tfrac{3}{2}x + \tfrac{1}{2}z\big)^3 dz \Big) dx = \int_0^2 \tfrac{1}{4}\big(3 - \tfrac{3}{2}x + \tfrac{1}{2}z\big)^4 \cdot 2 \Big|_{z=3x-6}^{z=0} dx$$

$$= \frac{1}{2}\int_0^2 \big(3 - \tfrac{3}{2}x\big)^4 dx = \frac{1}{2}\cdot\frac{1}{5}\big(3 - \tfrac{3}{2}x\big)^5 \cdot \big(-\tfrac{2}{3}\big)\Big|_0^2 = \frac{1}{15}\,3^5 = \frac{3^4}{5} = \frac{81}{5}$$

$$\iiint_A 3z \; dx\,dy\,dz$$



$y = 3 - \frac{3}{2}x$

$z = 3x + 2y - 6.$

$$= \int_0^2 \Big(\int_0^{3-\frac{3}{2}x} \Big(\int_{3x+2y-6}^{0} 3z \, dz \Big) dy \Big) dx = -\frac{3}{2}\int_0^2 \Big(\int_0^{3-\frac{3}{2}x} (3x+2y-6)^2 \, dy \Big) dx$$

$$= -\frac{3}{2}\int_0^2 \frac{1}{3}(3x+2y-6)^3 \cdot \frac{1}{2} \Big|_{y=0}^{y=3-\frac{3}{2}x} dx = +\frac{1}{4}\int_0^2 (3x-6)^3 dx = \frac{1}{16}\cdot\frac{1}{3}(3x-6)^4 \Big|_0^2$$

$$= -\frac{1}{48}\cdot(-6)^4 = -\frac{6^4}{48} = -\frac{6^3}{8} = -\frac{216}{8} = -27$$

Totalt $\frac{81}{5} - (-27) = \frac{216}{5}$

6.10.

Sylinderkoordinater

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$$\iiint_A f\,dx\,dy\,dz = \iiint_D f(r\cos\theta, r\sin\theta, z)\,\underset{\uparrow \text{NB!}}{r}\,dr\,d\theta\,dz$$

D = beskrivelsen av A i sylinderkoordinater

Kulekoordinater

$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$

$$\iiint_A f\,dx\,dy\,dz = \iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\underbrace{\rho^2\sin\phi}_{\uparrow \text{NB!}}\,d\rho\,d\phi\,d\theta$$

D = beskrivelsen av A i kulekoordinater

6.10.1 Bruk sylinderkoordinater

a) $\iiint_A x\,dx\,dy\,dz \qquad A = \{(x,y,z);\ x,y \geq 0 \quad x^2+y^2 \leq 9,\ 0 \leq z \leq 2$

$$D = \underset{r}{[0,3]} \times \underset{\theta}{[0,\tfrac{\pi}{2}]} \times \underset{z}{[0,2]}$$

$$= \int_0^3 \left(\int_0^{\frac{\pi}{2}} \left(\int_0^2 r^2\cos\theta\,dz \right) d\theta \right) dr = \int_0^3 \left(\int_0^{\frac{\pi}{2}} 2r^2\cos\theta\,d\theta \right) dr$$

$$= \int_0^3 2r^2\,dr = \frac{2}{3}r^3 \Big|_0^3 = 18$$

b) $\iiint_A xy\,dx\,dy\,dz \qquad A = \{(x,y,z);\ x^2+y^2 \leq 1,\ 0 \leq z \leq 4-x-y\}.$

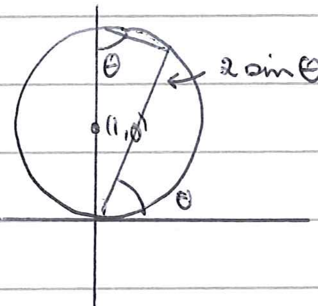
$$D = \{(r,\theta,z);\ 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 4-r(\sin\theta+\cos\theta)$$

$$= \int_0^1 \left(\int_0^{2\pi} \left(\int_0^{4-r(\cos\theta+\sin\theta)} r^3\cos\theta\sin\theta\,dz \right) d\theta \right) dr \quad =$$

$$\int_0^1 \left(\int_0^{2\pi} 4r^3 \cos\theta \sin\theta - r^4 \cos^2\theta \sin\theta - r^4 \cos\theta \sin^2\theta \, d\theta \right) dr$$

$$\int_0^1 4r^3 \sin^2\theta + \frac{1}{3} r^4 \cos^3\theta - \frac{1}{3} r^4 \sin^3\theta \Big|_0^{2\pi} dr = 0.$$

c) $\iiint_A z\sqrt{x^2+y^2}\, dx\, dy\, dz$ $\qquad A = \{(x,y,z);\ x^2+(y-1)^2 \leq 1,\ 0 \leq z \leq 2\}.$



$$= \int_0^{\pi} \left(\int_0^{2\sin\theta} \left(\int_0^2 z\, r^2\, dz \right) dr \right) d\theta$$

$$= \int_0^{\pi} \left(\int_0^{2\sin\theta} 2r^2\, dr \right) d\theta$$

$D = \{(r,\theta,z);\ 0 \leq \theta \leq \pi,\ 0 \leq r \leq 2\sin\theta,\ 0 \leq z \leq 2\}$

$$= \int_0^{\pi} \frac{2}{3} r^3 \Big|_0^{2\sin\theta} d\theta = \int_0^{\pi} \frac{16}{3} \sin^3\theta\, d\theta = \frac{16}{3} \int_0^{\pi} \sin\theta\,(1-\cos^2\theta)\, d\theta =$$

$$\frac{16}{3}\left(-\cos\theta + \frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi} = \frac{16}{3}\left(1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right) = \frac{16}{3} \cdot \frac{4}{3} = \underline{\frac{64}{9}}$$

6.10.2 Kulekoordinater, $x = \rho \sin\phi \cos\theta$ $\qquad z = \rho \cos\phi$
$y = \rho \sin\phi \sin\theta$

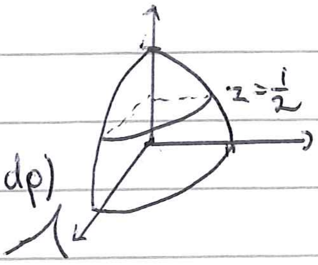
a) $\iiint_A (x^2+y^2)\, dx\, dy\, dz$ $\qquad$ A = kule om origo med radius 1.

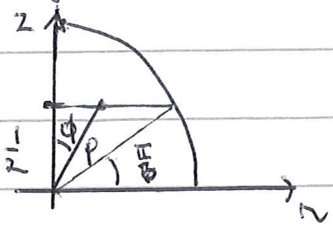
$D = \{(\rho,\theta,\varphi);\ 0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$

$$= \int_0^1 \left(\int_0^{2\pi} \left(\int_0^{\pi} \rho^2 \sin^2\phi \cdot \rho^2 \sin\phi\, d\phi \right) d\theta \right) d\rho$$

$$= \int_0^1 \rho^4\, d\rho \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin^3\phi\, d\phi = \frac{1}{5} \cdot 2\pi \cdot \frac{4}{3} = \underline{\underline{\frac{8\pi}{15}}}$$

b) $\iiint_A x\,dx\,dy\,dz$ $\qquad A=\{(x,y,z)\,;\ x,y\geq 0,\ z\geq \frac{1}{2},\ x^2+y^2+z^2\leq 1\}$





$$= \int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{3}}\left(\int_{\frac{1}{2\cos\phi}}^{1} \rho\sin\phi\cos\theta\cdot\rho^2\sin\phi\,d\rho\right)d\phi\,d\theta\right.$$

$\cos\phi = \frac{\frac{1}{2}}{\rho} \Rightarrow \rho = \frac{1}{2\cos\phi}$

$$= \int_0^{\frac{\pi}{3}}\left(\int_{\frac{1}{2\cos\phi}}^{1} \rho^3\sin^2\phi\,d\rho\right)d\theta = \int_0^{\frac{\pi}{3}}\left(\frac{1}{4}\rho^4\sin^2\phi\right)\Big|_{\rho=\frac{1}{2\cos\phi}}^{\rho=1}$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{3}} \sin^2\phi - \frac{\sin^2\phi}{16\cos^4\phi}\,d\phi = \frac{1}{4}\int_0^{\frac{\pi}{3}} \frac{1}{2} - \frac{1}{2}\cos 2\phi - \frac{1}{16}\underbrace{\sin\phi}_{u}\cdot\underbrace{\frac{\sin\phi}{\cos^4\phi}}_{v'}$$

$u' = \cos\phi \qquad v = \frac{1}{3}\cos^{-3}\phi$

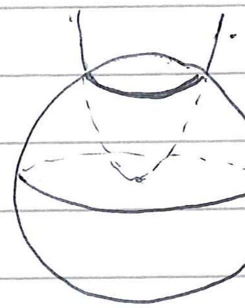
$$= \frac{1}{4}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\phi - \frac{1}{16}\left(\frac{1}{3}\frac{\sin\phi}{\cos^3\phi} - \int\frac{1}{3}\frac{1}{\cos^2\phi}\,d\phi\right)\right)\Bigg|_0^{\frac{\pi}{3}}$$

$$= \left(\frac{1}{8}\phi - \frac{1}{16}\sin 2\phi - \frac{1}{192}\frac{\sin\phi}{\cos^3\phi} + \frac{1}{192}\tan\phi\right)\Bigg|_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{24} - \frac{1}{16}\left(\frac{1}{2}\sqrt{3}\right) - \frac{1}{192}\cdot\frac{\frac{1}{2}\sqrt{3}}{\left(\frac{1}{8}\right)} + \frac{1}{192}\sqrt{3} = \frac{\pi}{24} - \frac{1}{32}\sqrt{3} - \frac{1}{48}\sqrt{3} + \frac{1}{192}\sqrt{3} = \underline{\underline{\frac{\pi}{24} - \frac{3\sqrt{3}}{64}}}$$

3 a $\iiint_A z\,dx\,dy\,dz$

A = området over $z = x^2+y^2$ og under $x^2+y^2+z^2 = 2$



Skjæren hverandre når

$z^2 + z = 2$

$z = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$

Altså $z = 1$.

(Sylinder)

$$= \int_0^{2\pi}\left(\int_0^1\left(\int_{r^2}^{\sqrt{2-r^2}} z\,dz\right)r\,dr\,d\theta\right.$$

$$= 2\pi\int_0^1 \left(\frac{1}{2}z^2\right)\Big|_{r^2}^{\sqrt{2-r^2}} r\,dr$$

$$= \pi\int_0^1 (2-r^2-r^4)\,r\,dr = \pi\int_0^1 2r - r^3 - r^5\,dr = \pi\left(2\cdot\frac{1}{2} - \frac{1}{4} - \frac{1}{6}\right) = \pi\left(\frac{12-3-2}{12}\right) = \frac{7}{12}\pi$$

c) $\iiint_A e^{-\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$ A = kule med senterum i origo, radius 1

(Kulekoord)

$$= \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{1} e^{-\rho}\rho^2 \sin\phi \, d\rho\right)d\phi\right)d\theta = \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\phi\, d\phi \cdot \int_0^1 \rho^2 e^{-\rho}\, d\rho$$

$$= 2\pi \cdot 2\left((-\rho^2 - 2\rho - 2)e^{-\rho}\right)\Big|_0^1 = \underline{4\pi\left(2 - \frac{5}{e}\right)}$$

d) $\iiint_A \sqrt{x^2+y^2}\,dx\,dy\,dz$ A området inni $x^2+y^2=1$

mellom xy-planet og flaten $z = (x^2+y^2)^{3/2}$
$= r^3$

(Sylinder)

$$= \int_0^{2\pi}\left(\int_0^{1}\left(\int_0^{r^3} r\, r\, dz\right)dr\right)d\theta = 2\pi\int_0^1 r^5\, dr = \underline{\underline{\frac{\pi}{3}}}.$$

5. $\iiint_R \sqrt{x^2+y^2}\,dx\,dy\,dz$ R = kula om 0 med radius 2

(Kulekoord.)

$$\int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{2} \rho \sin\phi \, \rho^2 \sin\phi\, d\rho\right)d\phi\right)d\theta = \int_0^{2\pi} d\theta \cdot \int_0^{\pi}\sin^2\phi\, d\phi \cdot \int_0^2 \rho^3\, d\rho$$

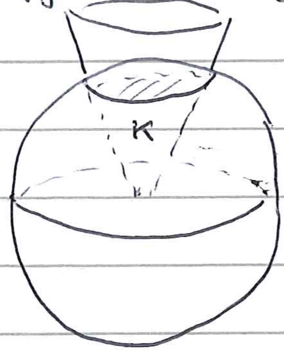
$$= 2\pi \cdot \left(\tfrac{1}{2}\pi\right)\cdot \tfrac{1}{4}\cdot 2^4 = \underline{\underline{4\pi^2}}$$

6.11.1 Volum av en kule med radius R

$$V = \iiint_K dx\,dy\,dz = \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{R} \rho^2 \sin\phi\, d\rho\right)d\phi\right)d\theta = \int_0^{2\pi} d\theta \cdot \int_0^{\pi}\sin\phi\, d\phi \int_0^R \rho^2\, d\rho$$

$$= 2\pi \cdot 2 \cdot \tfrac{1}{3}R^3 = \underline{\underline{\tfrac{4}{3}\pi R^3}}.$$

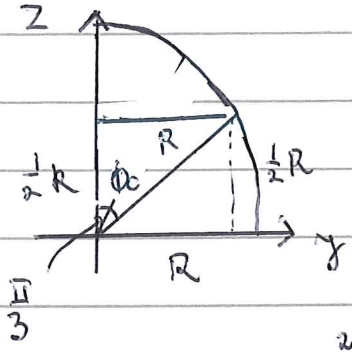
3. Volum av den delen av kulen $x^2+y^2+z^2 \le R^2$ som ligger over kjeglen $z = \sqrt{\frac{x^2+y^2}{3}}$



Skjæres hverandre når $x^2+y^2+\frac{1}{3}(x^2+y^2) = R^2$

dvs. $x^2+y^2 = \frac{3}{4}R^2$, sirkel med radius $\frac{1}{2}\sqrt{3}\,R$.

Vi får $z = \sqrt{\frac{1}{3}\cdot\frac{3}{4}R^2} = \frac{1}{2}R$



Vi har $0 \le \rho$ $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{3}$

siden $\cos\phi_0 = \frac{\frac{1}{2}R}{R} = \frac{1}{2}$.

$$V = \iiint_K dx\,dy\,dz = \int_0^{2\pi}\left(\int_0^{\frac{\pi}{3}}\left(\int_0^R \rho^2 \sin\phi \, d\rho\right) d\phi\right) d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{3}} \left(\frac{1}{3}\rho^3 \sin\phi\right)\Big|_0^R = \frac{2}{3}\pi R^3 \int_0^{\frac{\pi}{3}} \sin\phi \, d\phi = \frac{2}{3}\pi R^3 \left(-\cos\phi\right)\Big|_0^{\frac{\pi}{3}}$$

$$= \frac{2}{3}\pi R^3\left(-\frac{1}{2} - (-1)\right) = \underline{\underline{\frac{\pi}{3}R^3}}$$

6. Massen til sylinderen $x^2+y^2 \le 1$, $0 \le z \le 1$, tetthet $f(x,y,z) = \frac{1}{x^2+y^2+z^2}$

$$M = \iiint_S \frac{dx\,dy\,dz}{x^2+y^2+z^2} \underset{\text{Sylinderk.}}{=} \int_0^{2\pi}\left(\int_0^1\left(\int_0^1 \frac{r\,dz}{r^2+z^2}\right) dr\right) d\theta$$

$$= 2\pi \int_0^1\left(\int_0^1 \frac{r\,dz}{r^2+z^2}\right) dr = 2\pi \int_0^1\left(\int_0^1 \frac{(1/r)\,dz}{1+\left(\frac{z}{r}\right)^2}\right) dr = 2\pi \int_0^1 \left(\arctan\frac{z}{r}\right)\Big|_0^1 dr$$

$$= 2\pi \int_0^1 \arctan\frac{1}{r}\, dr = 2\pi\left(r\arctan\frac{1}{r}\Big|_0^1 + \int_0^1 \frac{r\,dr}{1+r^2}\right) = 2\pi\left(\frac{\pi}{4} + \frac{1}{2}\ln(1+r^2)\Big|_0^1\right)$$

$\left(u = \arctan\frac{1}{r} \quad u' = \frac{1}{1+(1/r)^2}\cdot\left(-\frac{1}{r^2}\right) = \frac{-1}{1+r^2} \quad v' = 1,\ v = r\right)$ $= \frac{\pi^2}{2} + \pi\ln 2$