12.6.1 Konvergensintervallet

a) $\sum_{n=0}^{\infty} (x-2)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(x-2)^{n+1}}{(x-2)^n}\right| = |x-2|$$

Konvergent når $|x-2|<1 \Leftrightarrow x\in(1,3)$.
$x=3$ gir $\sum 1^n$, divergent ved divergenstesten DT
$x=1$ gir $\sum (-1)^n$, divergent ved DT

b) $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ $\quad \lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{x^{n+1}\cdot 3^n}{3^{n+1}\cdot x^n}\right| = \frac{|x|}{3}$

Konvergent når $|x|<3 \Leftrightarrow x\in(-3,3)$
$x=\pm 3$ er begge divergente, som i a).

c) $\sum_{n=0}^{\infty} n(2x-1)^n$ $\quad \lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(n+1)(2x-1)^{n+1}}{n(2x-1)^n}\right|$

$$= \lim \frac{n+1}{n}|2x-1| = |2x-1|$$

Konvergent når $|2x-1|<1 \Leftrightarrow |x-\frac{1}{2}|<\frac{1}{2} \Leftrightarrow x\in(0,1)$
$x=0$ gir $\sum (-1)^n n$, Divergent ved DT
$x=1$ gir $\sum n$. Divergent ved DT.

d) $\sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}}$ $\quad \lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(x+1)^{n+1}\cdot\sqrt{n}}{\sqrt{n+1}\cdot(x+1)^n}\right|$

$$= \lim\sqrt{\frac{n}{n+1}}\,|x+1| = |x+1|$$

Konvergent når $|x+1|<1 \Leftrightarrow x\in(-2,0)$.
$x=-2$ gir $\sum (-1)^n/\sqrt{n}$, konvergent alternerende rekke.
$x=0$ gir $\sum 1/\sqrt{n}$, divergent. Konvergensintervall $[-2,0)$

e) $\sum_{n=2}^{\infty} \frac{(1/4)^n (x-1)^n}{n(n-1)}$

$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(1/4)^{n+1} (x-1)^{n+1} \quad n(n-1)}{(n+1)n \cdot (1/4)^n (x-1)^n}\right|$

$= \lim \frac{1}{4} \frac{n-1}{n+1} \cdot |x-1| = \frac{1}{4}|x-1|$

Konvergent når $|x-1| < 4 \Leftrightarrow x \in (-3, 5)$.

$x = -3$ gir $\sum (-1)^n / n(n-1)$, konvergent

$x = 5$ gir $\sum 1/n(n-1)$, konvergent.

Konvergensintervall $[-3, 5]$.

g) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$

$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)!\,(n+1)!\, x^{n+1}\,(2n)!}{(2n+2)!\; n!\, n!\, x^n}\right|$

$= \lim \frac{(n+1)(n+1)}{(2n+1)(2n+2)} |x| = \frac{1}{4}|x|$. Konvergent når $|x| < 4$.

For $x = 4$ er $a_n = \frac{(1\cdot 2 \cdot \cdots \cdot n)\,(1 \cdot 2 \cdot \cdots \cdot n) \cdot 2^{2n}}{1 \cdot 2 \cdot \cdots \cdot 2n} = \frac{(2\cdot 4 \cdot \cdots \cdot 2n)(2 \cdot 4 \cdot \cdots \cdot 2n)}{1 \cdot 2 \cdot \cdots \cdot 2n}$

$= \frac{2 \cdot 4 \cdot \cdots \cdot 2n}{1 \cdot 3 \cdot \cdots \cdot 2n-1} > 1$, altså divergent ved DT. Samme for $x = -4$.

12.7. Vi bruker her bare at potensrekken kan deriveres og integreres leddvis i det åpne konvergensintervallet.

12.7.1

a) $f(x) = \sum_{n=0}^{\infty} n^2 x^n$, $f'(x) = \sum_{n=1}^{\infty} n^2 \cdot n x^{n-1} = \sum_{n=1}^{\infty} n^3 x^{n-1}$

$= \sum_{n=0}^{\infty} (n+1)^3 x^n$.

$$F(x) = \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \int_0^x n^2 t^n\,dt = \sum_{n=0}^{\infty} \frac{n^2}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(n-1)^2}{n} x^n$$

b) $$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n}{n+1} x^{n-1} = \sum_{n=0}^{\infty} \frac{n+1}{n+2} x^n$$

$$F(x) = \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \int_0^x \frac{t^n}{n+1}\,dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

12.7.2 Vi vet at den geometriske rekka $a + at + at^2 + \dots = \sum_{n=0}^{\infty} at^n$ konvergerer for $|t| < 1$ og har sum $\frac{a}{1-t}$.

a) Setter vi $t = x^2$, $a = 1$, får vi at $\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$

og at det konvergerer når $|x^2| < 1 \Leftrightarrow |x| < 1$.

b) Vi integrerer hver side

$$\int_0^x \frac{1}{1-t^2}\,dt = \int_0^x \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt = \frac{1}{2}(\ln|1+x| - \ln|1-x|)$$
$$= \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right|$$

$$\int_0^x \sum_{n=0}^{\infty} t^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$ når $|x| < 1$.

Altså

$$\ln\left|\frac{1+x}{1-x}\right| = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2m+1}$$ for $|x| < 1$.

c) Vi setter $x = \frac{1}{2}$ ligningen over. Venstre side $\ln\left|\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right| = \ln 3$

Høyre side: $2\sum_{n=0}^{\infty} \frac{1}{(2n+1)} \cdot \left(\frac{1}{2}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)\, 2^{2n}}$

Dette viser c).

12.7.3. a) Setter vi $a = x^2$ og $t = x^3$ får vi

$$\sum_{n=0}^{\infty} x^{3n+2} = \sum_{n=0}^{\infty} x^2 (x^3)^n = \frac{x^2}{1-x^3}$$ når $|x^3| < 1 \Leftrightarrow |x| < 1$.

b) Vi integrerer hver side. Anta $|x| < 1$

$$\int_0^x \frac{t^2}{1-t^3}\,dt = \int_0^{x^3} \frac{\frac{1}{3}du}{1-u} = -\frac{1}{3}\ln|1-u|\Big|_0^{x^3} = -\frac{1}{3}\ln(1-x^3)$$

$u = t^3$

$du = 3t^2\,dt$

$$\int_0^x \sum_{n=0}^{\infty} t^{3n+2}\,dt = \sum_{n=0}^{\infty} \frac{x^{3n+3}}{3n+3} = \sum_{n=1}^{\infty} \frac{x^{3n}}{3n}$$

Dette gir

$$\ln(1-x^3) = -\sum_{n=1}^{\infty} \frac{x^{3n}}{n} \quad \text{for } |x| < 1.$$

c) For $x = -1$ konvergerer rekka $\sum_{n=1}^{\infty} \frac{x^{3n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{3n}}{n}$

$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, den alternerende harmoniske rekka.

Abels teorem gir da at vi kan sette inn $x = -1$ i ligningen, dvs

$$\ln 2 = -\sum_{n=1}^{\infty} \frac{(-1)^{3n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{3n+1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

12.8.1

a) Taylor-rekka til $f(x) = e^x$ i punktet 1.

Vi har $f^{(k)}(x) = e^x$, så $f^{(k)}(1) = e$ for alle k og

$$Tf(x) = \sum_{k=0}^{\infty} \frac{e}{k!}(x-1)^k$$

Konvergensintervall:

$$\lim \left|\frac{a_{k+1}}{a_k}\right| = \lim \left|\frac{e\,(x-1)^{k+1}\,k!}{(k+1)!\,e\,(x-1)^k}\right| = \lim \frac{|x-1|}{k+1} = 0$$

Rekka konvergerer for alle x, dvs. $I = (-\infty, \infty)$

For å vise at rekka konvergerer mot $f(x)$, må vi vise at feilleddet går mot 0.

Gi ett x, la $M = \max\{|f^{(n+1)}(t)|;\ t \text{ mellom } 1 \text{ og } x\}$.

Lett å se at

$$M = \max\{e^x, e\}$$

Vi har da

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-1|^{n+1} \to 0 \text{ når } n \to \infty.$$

b) Taylor-rekka til $f(x) = \sin x$ i punktet $\pi/4$

$$f\left(\tfrac{\pi}{4}\right) = \tfrac{1}{2}\sqrt{2}$$

$$f'(x) = \cos x,\quad f'\left(\tfrac{\pi}{4}\right) = \tfrac{1}{2}\sqrt{2}$$

$$f''(x) = -\sin x,\quad f''\left(\tfrac{\pi}{4}\right) = -\tfrac{1}{2}\sqrt{2}$$

$$f'''(x) = -\cos x,\quad f'''\left(\tfrac{\pi}{4}\right) = -\tfrac{1}{2}\sqrt{2}$$

o.s.v.

$$Tf(x) = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}\left(x-\tfrac{\pi}{4}\right) - \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}\left(x-\tfrac{\pi}{4}\right)^2 - \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{6}\left(x-\tfrac{\pi}{4}\right)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{\left[\frac{n}{2}\right]}\sqrt{2}}{2\,n!}\left(x-\tfrac{\pi}{4}\right)^n$$

der $[a]$ = største hele tall $\leq a$.

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{|x-\frac{\pi}{4}|}{(n+1)} = 0$$

Rekka konvergerer for alle x, dvs. $I = (-\infty, \infty)$

Gitt x, sett

$$M = \max\{|f^{(n+1)}(t)|;\ t \text{ mellom } \tfrac{\pi}{4} \text{ og } x\}.$$

Klart at $M \leq 1$ siden de deriverte av f er $\pm \sin t$, $\pm \cos t$.

Dermed

$$|R_n(x)| \leq \frac{1}{(n+1)!}|x-\tfrac{\pi}{4}|^{n+1} \to 0 \quad \text{når } n \to \infty.$$

Dette viser at Taylor rekka konvergerer mot f.

c) Taylor-rekka til $f(x) = x^3 - 2x^2 + 7x - 4$ i punktet -1.

Vi har $f(-1) = -1 - 2 \cdot 1 - 7 - 4 = -14$

$f'(x) = 3x^2 - 4x + 7$, $f'(-1) = 3 - 4\cdot(-1) + 7 = 14$

$f''(x) = 6x - 4$, $f''(-1) = -6 - 4 = -10$

$f'''(x) = 6$, $f'''(-1) = 6$

$f^{(4)}(x) = 0$

$$Tf(x) = -14 + 14(x+1) - \frac{10}{2}(x+1)^2 + \frac{6}{6}(x+1)^3$$

$$= -14 + 14(x+1) - 5(x+1)^2 + (x+1)^3$$

Tf er en endelig rekke. Konvergerer overalt og $R_n(x) = 0$ når $n \geq 3$, altså $Tf(x) = f(x)$.

12.8.3 a) $g(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for alle x.

$$f(x) = \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+2}$$

b) $g(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for alle x

$f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

c) $g(x) = \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for alle x

$f(x) = \cos 3x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(3x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n}}{(2n)!} x^{4n}$

12.8.13 a) $\sum_{n=1}^{\infty} n x^n$ konvergerer hvor?

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{(n+1)x^{n+1}}{n x^n} \right| = \lim \frac{n+1}{n} |x| = |x|$

Konvergerer for $|x| < 1$ dvs. $x \in (-1, 1)$.
Divergerer i begge endepunktene ved DT.

b) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$. Leddvis derivasjon gir

$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$ så $\sum_{n=1}^{\infty} n x^n = \underline{\underline{\frac{x}{(1-x)^2}}}$

12.8.15 a) Konvergensområdet til $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}$?

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{x^{2n+2} (2n+1)}{(2n+3)\, x^{2n}} \right| = \lim \frac{2n+1}{2n+3} |x^2| = |x^2|$

Konvergerer for $|x^2| < 1 \Leftrightarrow |x| < 1$.
Endepunktene: $x = \pm 1$ gir $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ konvergent alternerende rekke, siden

$\frac{1}{2n+1}$ går monotont mot 0 når $n \to \infty$.

Konvergensområde: $\underline{\underline{[-1, 1]}}$

b) $\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1+x^2}$ for $|x|<1$, geometrisk

rekke. Integrasjon gir

$$(*) \quad \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = \int_0^x \frac{dt}{1+t^2} = \arctan x$$

Hvis $x \neq 0$ kan vi dele

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2n+1} = \underline{\underline{\frac{\arctan x}{x}}} \qquad \text{for } x \neq 0.$$

For $x=0$ er summen lik 1.

c) $(*)$ gir $\arctan\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{2}\right)^{2n+1}}{(2n+1)} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^{2n+1}\cdot(2n+1)}$

$$= \frac{1}{2\cdot 1} - \frac{1}{8\cdot 3} + \frac{1}{32\cdot 5} - \ldots$$

en alternerende rekke. Feilen er da mindre enn neste ledd. Siden $32\cdot 5 = 160 > 100$, er det nok å ta med 2 ledd, dvs.

$$\arctan\frac{1}{2} \approx \frac{1}{2} - \frac{1}{24} = \underline{\underline{\frac{11}{24}}}$$

12.8.17 $\sum_{n=1}^{\infty}\frac{n}{2^n}(x-1)^n$ konvergerer hvor?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(n+1)(x-1)^{n+1}\cdot 2^n}{2^{n+1}\, n\,(x-1)^n}\right| = \lim \frac{n+1}{2n}|x-1| = \frac{1}{2}|x-1|$$

Konvergerer når $|x-1|<2$, dvs $x\in(-1,3)$.
Endepunktene: $x=-1$ gir $\sum_{n=1}^{\infty}(-1)^n n$ og $x=1$ gir $\sum_{n=1}^{\infty} n$, begge divergente ved DT.

Sum?

Vi har $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n = \frac{1}{1-\frac{x-1}{2}} = \frac{2}{3-x}$

når $|x-1|<2$ (geometrisk rekke). Derivasjon gir

$$\sum_{n=1}^{\infty} \frac{n}{2^n}(x-1)^{n-1} = \frac{2}{(3-x)^2}$$

. Multipliserer med $x-1$:

$$\sum_{n=1}^{\infty} \frac{n}{2^n}(x-1)^n = \underline{\underline{\frac{2(x-1)}{(3-x)^2}}}$$