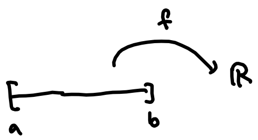
Regnelørdag: VB aud 1, kl. 10:00 - 16:00.

Greens teorem (seksjon 6.5)

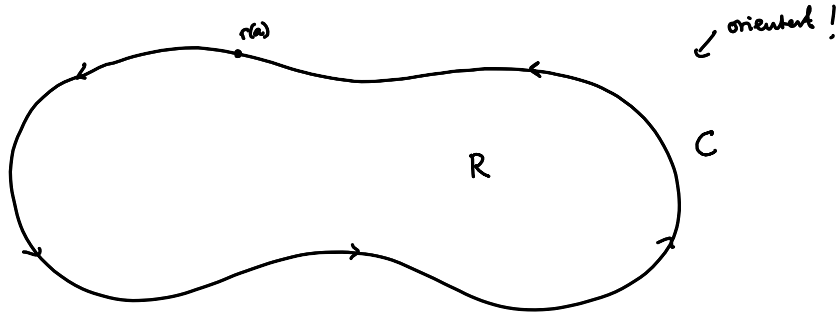
Motivasjon: Analysens fundamentalteorem



$\rightsquigarrow \quad \int_a^b f'(x)\,dx = f(b) - f(a)$

∴ integralet til en derivert er bestemt av verdiene til f i endepunktene.

I $\mathbb{R}^2$: Se på en kurve C, parametrisert ved $\mathbf{r} : [a,b] \longrightarrow \mathbb{R}^2$



Antagelser:

- C er enkel: $(\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for t_1, t_2 forskjellige i $(a,b))$
  (krysser ikke seg selv)

- C er lukket:
  $\mathbf{r}(a) = \mathbf{r}(b)$
- R = området avgrenset av C
- $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ vektorfelt på C
  Anta at P, Q kontinuerlige deriverbare i en kule som inneholder R.

Vi vil regne ut $\int_C \mathbf{F}\cdot d\mathbf{r}$:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt$$

$$= \int_a^b (P(x(t),y(t)), Q(x(t),y(t)))\cdot(x'(t), y'(t))\,dt$$

$$= \int_a^b P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\;dt$$

Skriver ofte $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy$ der $dx = x'(t)\,dt$, $dy = y'(t)\,dt$

Greens teorem

Anta at C er orientert <u>mot</u> klokken (= "positiv omløpsretning").

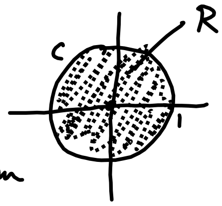
Da gjelder

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

husk denne som $\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}$

Denne formelen kan brukes både til å regne ut linjeintegraler og dobbeltintegraler.

<u>Eks 1</u> La $P(x,y) = 4y$, $Q(x,y) = 5x$ $\quad R = B(0,1)$, $\quad C = \partial R =$ enhetssirkelen



parametrisering av C: $\quad \mathbf{r}(t) = (\cos t, \sin t) \qquad t \in [0, 2\pi]$

$$\frac{\partial Q}{\partial x} = 5 \qquad \frac{\partial P}{\partial y} = 4$$

$$\rightsquigarrow \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_R (5-4)\,dx\,dy = \text{areal}(R) = \underline{\pi}.$$

$$dx = x'(t)\,dt = -\sin t \cdot dt$$

$$dy = y'(t)\,dt = \cos t \cdot dt$$

$$\int_C P\,dx + Q\,dy = \int_0^{2\pi} (4 \cdot \sin t)(-\sin t)\,dt + 5(\cos t)\cos t\,dt$$

$$= \int_0^{2\pi} 5\cos^2 t - 4\sin^2 t\,dt$$

$$= \int_0^{2\pi} (5 - 9\sin^2 t)\,dt$$

$5 = 5\cos^2 t + 5\sin^2 t$

$5 - 9\sin^2 t = 5\cos^2 t - 4\sin^2 t$

$$= \frac{1}{2}\Big[t + 9 \sin t \cdot \cos t \Big]_0^{2\pi} = \frac{1}{2} \cdot 2\pi = \underline{\pi}$$

Eks 2   Finn $\int_C \mathbf{F}\cdot d\mathbf{r}$ der $\mathbf{F}(x,y) = \left(\sqrt{1+e^{x^2}}+2y,\ 4x - e^{y^2}\sin y\right)$

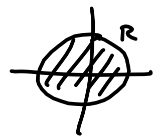
der C er sirkelen $x^2+y^2=9$ orientert mot klokken.

$$P = \sqrt{1+e^{x^2}} + 2y$$
$$Q = 4x - e^{y^2}\sin y$$
$$\rightsquigarrow \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (4) - (2) = 2$$

$$\rightsquigarrow \int_C \mathbf{F}\cdot d\mathbf{r} \overset{\text{Green}}{=} \iint_R 2 \, dx\, dy$$

$$= 2\cdot \text{areal}(R) = 2\cdot\pi\cdot 3^2 = \underline{18\pi}$$

Eks 3   Finn arealet til området begrenset av ellipsen $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. $\leftarrow C$



parametrisering av C: $\mathbf{r}(t) = (a\cos t,\ b\sin t)$ $t\in[0,2\pi]$

(mot klokken ✓)

Kan vi finne P, Q s.a.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \ ?$$

I så fall vil

$$\int_C P\,dx + Q\,dy = \iint 1 \, dx\, dy = \text{areal}(R)$$

Ja: kan f.eks velge $Q = x$, $P = 0$ (eller $Q = 0$, $P = -y$)

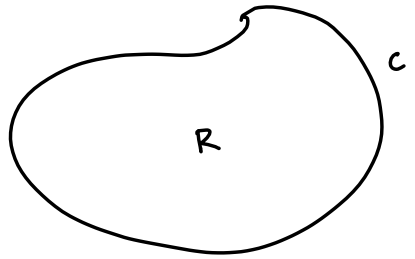
$$\text{areal}(R) \overset{\text{Green}}{=} \int_C x\, dy = \int_0^{2\pi} (a\cos t)(b\cos t)\, dt$$

$dy = y'(t)\,dt$
$= b\cos t\, dt$

$$= ab\int_0^{2\pi} \cos^2 t\, dt$$

$$= \frac{ab}{2}\Big[t + \sin t\cdot\cos t\Big]_0^{2\pi} = \underline{\pi ab}$$

Korollar av Greens teorem



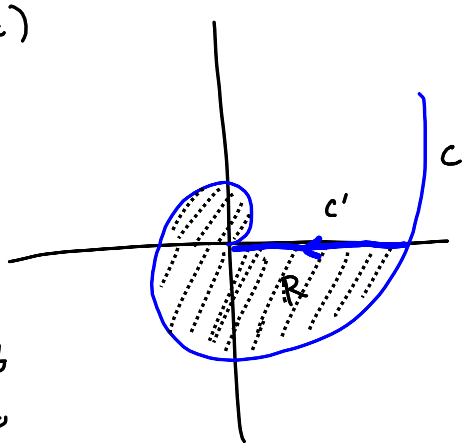
C = enkel, lukket, stykkevis glatt kurve orientert mot klokken

R = området avgrenset av C

$$\rightsquigarrow \quad \text{areal}(R) = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C -y\,dx + x\,dy.$$

Eks Finn arealet begrenset av kurven C gitt ved

$$\mathbf{r}(t) = (t\cdot\cos t,\ t\cdot\sin t)$$
$$t \in [0, 2\pi]$$



C orientert mot klokken ✓

prøver $P = -\frac{1}{2}y$; Greens teorem.
$Q = \frac{1}{2}x$

$x(t) = t\cos t \quad \rightsquigarrow \quad dx = (-t\sin t + \cos t)\,dt$
$y(t) = t\sin t \qquad\quad dy = (t\cos t + \sin t)\,dt$

$$\text{areal}(R) = \frac{1}{2}\int_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_C (t\cos t)(t\cos t + \sin t)\,dt - (t\sin t)(-t\sin t + \cos t)\,dt$$

$$= \frac{1}{2}\int_C t^2\underbrace{(\cos^2 t + \sin^2 t)}_{=1} + t\,(\underbrace{\cos t\sin t - \cos t\sin t}_{=0})\,dt$$

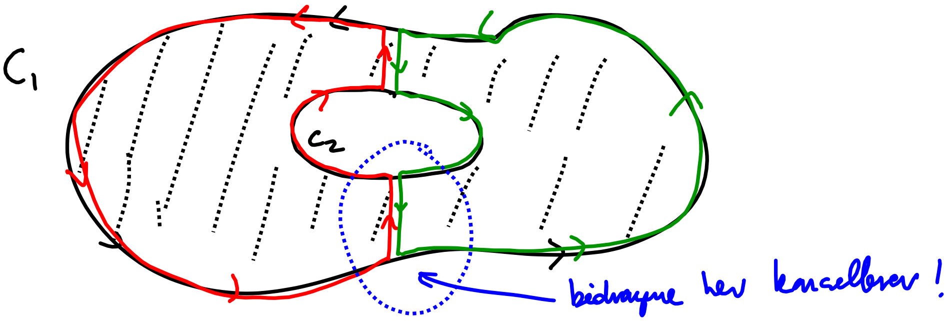
$$= \frac{1}{2}\int_0^{2\pi} t^2\,dt = \frac{1}{2}\,\frac{t^3}{3}\Big|_0^{2\pi} = \underline{\frac{4\pi^3}{3}}$$

Integralet langs C' er 0, da $y=0$ og $dy=0$

$$\rightsquigarrow \quad \int_{C'} x\,\underset{0}{\underset{\shortparallel}{dy}} - y\,\underset{0}{\underset{\shortparallel}{dx}} = 0.$$

Oppdeling av områder

Kan også ha områder R med "hull":

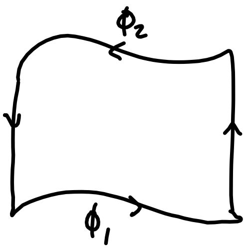


$$\rightsquigarrow \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \underbrace{\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}_{\uparrow \text{ vil finne denne}}$$

Idé for beviset for Greens teorem:

(I) Holder å vise Greens teorem for områder av type I og II (ved oppdeling).

(II) Vise Green for type I og II.



Bevis for type I områder (Type II helt likt)

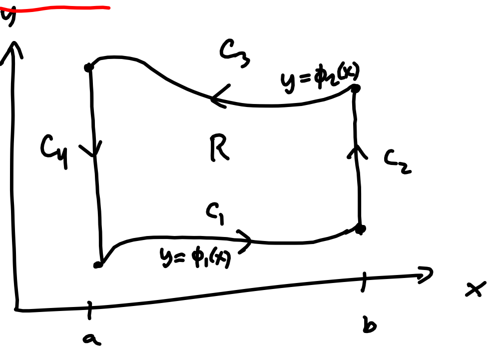
Parametriseringer:

C_1: $x = x$, $y = \phi_1(x)$, $x \in [a,b]$

C_2: $x = b$, $y = y$, $y \in [\phi_1(b), \phi_2(b)]$

C_3: $x = x$, $y = \phi_2(x)$, $x \in [a,b]$ (motsatt vei!)

C_4: $x = a$, $y = y$, $y \in [\phi_1(a), \phi_2(a)]$ (motsatt vei)



R: $x \in [a,b]$, $y \in [\phi_1(x), \phi_2(x)]$

Viser

(1) $$\int_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dx\,dy$$

(2) $$\int_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dx\,dy$$

⟶ legg sammen ⇝ Green.

(1): $$HS = \iint_R \left(-\frac{\partial P}{\partial y}\right) dx\,dy = -\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx$$

Analysens fundamentalteorem

$$= -\int_a^b P(x,\phi_2(x)) - P(x,\phi_1(x))\,dx$$

$$= \int_a^b P(x,\phi_1(x)) - P(x,\phi_2(x))\,dx$$

$$VS = \int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx + \int_{C_4} P\,dx$$

på C_2 og C_4 har vi $dx = x'(t)\,dt = 0$

$$= \int_{C_1} P\,dx + \int_{C_3} P\,dx$$

$$= \int_a^b P(x,\phi_1(x))\,dx - \int_a^b P(x,\phi_2(x))\,dx$$

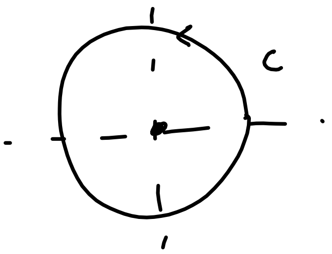
bytte fortegn for å få C_3 "riktig vei"!

$$= \int_a^b P(x,\phi_1(x)) - P(x,\phi_2(x))\,dx$$

⟶ (1) er OK.

Helt tilsvarende argument for (2) (bytte om P og Q).

Eks La
$$P = -\frac{y}{x^2+y^2}$$
$$Q = \frac{x}{x^2+y^2}$$

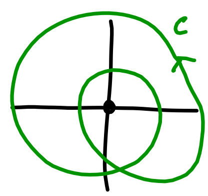


C: parametrisert ved $\mathbf{r}(t) = (\cos t, \sin t)$

$$\rightsquigarrow \int_C P\,dx + Q\,dy = \int_C \frac{-\sin t}{1}\overset{dx}{\overbrace{(-\sin t)\,dt}} + \frac{\cos t}{1}\overset{dy}{\overbrace{(\cos t)\,dt}}$$
$$= \int_C \sin^2 t + \cos^2 t \, dt = 2\pi$$

Prøver med Green:
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \left(\frac{1\cdot(x^2+y^2) - x(2x)}{(x^2+y^2)^2}\right) - \left(\frac{1\cdot(x^2+y^2) - 2y\cdot y}{(x^2+y^2)^2}\right)$$
$$= 0$$
$$\iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy = 0 \;\ldots$$

Merk: $\mathbf{F}$ er ikke definert i $(0,0)$!
(Men $\mathbf{F}$ er glatt langs kurven C).



[Interessant faktum: For $C \subset \mathbb{R}^2$ en kurve er
$$\frac{1}{2\pi}\int_C \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2} = \text{antall ganger } C \text{ sirkulerer origo.}$$
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