Generelt om multiple integraler i $\mathbb{R}^n$

Gitt en mengde $A \subset \mathbb{R}^n$ + en funksjon $f: A \to \mathbb{R}$ $\rightsquigarrow$ Skal definere

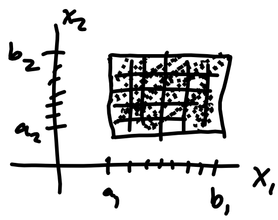
$$\int \cdots \int_A f(x_1, \dots, x_n)\, dx_1\, dx_2 \cdots dx_n$$

Teorien for dobbeltintegraler generaliseres uten store endringer til $\mathbb{R}^n$.

- Rektangler: mengder på formen

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n] = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; \begin{matrix} a_1 \le x_1 \le b_1 \\ \vdots \\ a_n \le x_n \le b_n \end{matrix} \right\}$$

$$|R| = \text{volumet til } R = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$



- Får partisjoner Π av R ved å dele opp $[a_1, b_1], \dots, [a_n, b_n]$

For en funksjon $f: R \to \mathbb{R}$ får vi trappesummer

$$N(\Pi) = \sum_{i=1}^{N} m_i\, |R_i| \qquad \Pi = \{R_1, \dots, R_N\}$$

$$m_i = \inf_{R_i} f(x) \qquad M_i = \sup_{R_i} f(x)$$

$$Ø(\Pi) = \sum_{i=1}^{N} M_i\, |R_i|$$

$\rightsquigarrow$ øvre/nedre integraler $\overline{\int \cdots \int} f\, dx_1 \cdots dx_n$ og $\underline{\int \cdots \int} f\, dx_1 \cdots dx_n$

Vi sier at f er <u>integrerbar</u> dersom disse to er like, og kaller verdien for det <u>multiple integralet</u>

$$\int \cdots \int_R f(x_1 \dots x_n)\, dx_1 \cdots dx_n .$$

Teoremene for $n=2$ tilfellet generaliserer:

- Kontinuerlige funksjoner er integrerbare
- $$\int_{[a_1,b_1]\times\cdots\times[a_n,b_n]}\!\!\cdots\int f(x_1..x_n)\,dx_1..dx_n = \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\cdots\left(\int_{a_n}^{b_n} f(x_1..x_n)\,dx_n\right)dx_{n-1}\right)..\,dx_1$$

  funksjon i x_n

  Integrasjonsrekkefølgen er villkårlig.

Dersom $S\subset\mathbb{R}^n$ er begrenset definerer vi

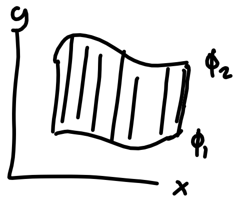
$$\int_S\cdots\int f\,dx_1..dx_n := \int_R\cdots\int f_S\,dx_1..dx_n$$

der R er et rektangel som inneholder S og $f_S(x) = \begin{cases} f(x) & x\in S \\ 0 & \text{ellers}\end{cases}$.

Trippelintegraler: $\underline{n=3}$. $f(x,y,z)$ funksjon i 3 variable

$$\longrightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz.$$

$\leadsto$ har analoger til type I og type II områder:



Setning $A\subset\mathbb{R}^2$ begrenset, Jordanmålbar mengde.

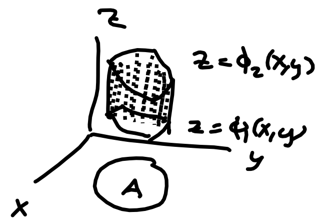
$\phi_1,\phi_2: A\longrightarrow\mathbb{R}$ kontinuerlige funksjoner s.a $\phi_2\geq\phi$ overalt.

La
$$S=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \begin{array}{c}(x,y)\in A\\ \phi_1(x,y)\leq z\leq\phi_2(x,y)\end{array}\right\}.$$

Da er

$$\iiint_S f(x,y,z)\,dx\,dy\,dz = \iint_A\left(\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\right)dx\,dy$$

funksjon i z



Eks $\quad R = [0,1] \times [0,1] \times [0,1] = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \\ 0 \le z \le 1 \end{array} \right\}$

$f(x,y,z) = x y^2 z^3$

Finn $I = \iiint_R f \, dx\, dy\, dz$.

$$I = \int_0^1 \left(\int_0^1 \left(\int_0^1 x y^2 z^3 \, dz \right) dy \right) dx$$

$$= \int_0^1 \int_0^1 \left[x y^2 \frac{z^4}{4} \right]_{z=0}^{z=1} dy\, dx$$

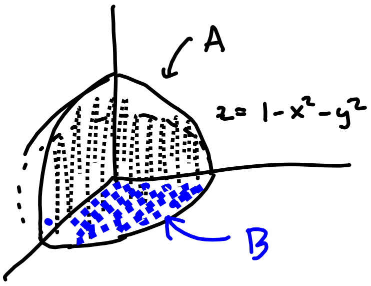
$$= \frac{1}{4} \int_0^1 \int_0^1 x y^2 \, dy\, dx$$

$$= \frac{1}{4} \int_0^1 x \frac{y^3}{3} \Big]_0^1 dx$$

$$= \frac{1}{4} \cdot \frac{1}{3} \int_0^1 x \, dx$$

$$= \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} = \underline{\frac{1}{24}}$$

Eks



$$A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le 1 - x^2 - y^2\}$$

Regn ut $I = \iiint_A y^2 \, dx\,dy\,dz$.

La $B = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$. ← sirkelskive av radius 1

$$\leadsto \quad A = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} (x,y) \in B \\ 0 \le z \le 1 - x^2 - y^2 \end{array} \right\}$$

parametriser B: $\quad x = r\cos\theta \quad \theta \in [0, 2\pi]$

$\qquad\qquad\qquad\quad y = r\sin\theta \quad r \in [0,1]$

$$\leadsto \quad I = \iint_B \left(\int_0^{1-x^2-y^2} y^2 \, dz \right) dx\,dy$$

$$= \iint_B (1 - x^2 - y^2)\, y^2 \, dx\,dy$$

$$= \iint_B (1 - r^2) \cdot r^2 \sin^2\theta \cdot \underline{r} \; dr\,d\theta$$

$$= \int_0^1 \int_0^{2\pi} (r^3 - r^5) \cdot \sin^2\theta \; d\theta\, dr$$

$$= \int_0^1 (r^3 - r^5) \underbrace{\left[\tfrac{1}{2}(\theta - \sin\theta) \right]_0^{2\pi}}_{\pi} dr$$

$$= \int_0^1 (r^3 - r^5)\,\pi \, dr$$

$$= \pi \left[\frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 = \pi\left(\frac{1}{4} - \frac{1}{6}\right) = \pi\left(\frac{2}{24}\right) = \underline{\frac{\pi}{12}}$$

Skifte av variabel : trippelintegraler:

La $T: \underset{u,v,w}{D} \longrightarrow \mathbb{R}^3$ være en parametrisering av en mengde $A = T(D)$

$x = x(u,v,w)$

$y = y(u,v,w)$

$z = z(u,v,w)$

Anta at $\det T'(u,v,w) \neq 0$ for alle $(u,v,w) \in D$.

$f: A \longrightarrow \mathbb{R}$ kontinuerlig funksjon.

Da er

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_D f(T(u,v,w)) \cdot \left|\det T'(u,v,w)\right| du\,dv\,dw$$

$$= \iiint_D f(T(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

determinanten til Jacobimatrisen til T.

Interessante spesialtilfeller:

- Sylinderkoordinater:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

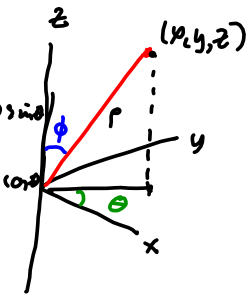
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = \underline{r}$$

- Kulekoordinater:

$$x = \rho\sin\phi\cdot\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

$$\phi \in [0,\pi] \qquad \theta \in [0,2\pi]$$

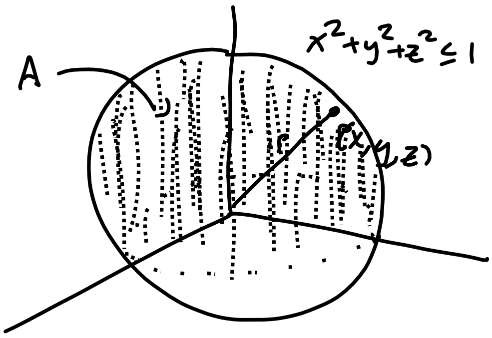


$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \ldots = \rho^2 \underline{\sin\phi}$$

Eks La $A = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} x^2+y^2+z^2 \leq 1 \\ z \geq 0 \end{array} \right\}$

Finn $I = \iiint_A z^2 \, dx\,dy\,dz$.



Bruker kulekoordinater: $\rho \in [0,1]$

$0 \leq \phi \leq \pi/2$

$0 \leq \theta \leq 2\pi$

$$\rightsquigarrow \quad I = \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \underset{\substack{\uparrow \\ z^2}}{(\rho \cos \phi)^2} \; \underset{\substack{\uparrow \\ |\det T'(\rho,\phi,\theta)|}}{(\rho^2 \sin \phi)} \, d\theta \, d\phi \, d\rho$$

$$= 2\pi \int_0^1 \int_0^{\pi/2} \rho^4 \cos^2 \phi \sin \phi \, d\phi \, d\rho$$

$$= 2\pi \int_0^{\pi/2} \cos^2\phi \sin\phi \; \underbrace{\left[\frac{\rho^5}{5} \right]_0^1}_{1/5} \, d\phi$$

$$= \frac{2\pi}{5} \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi \qquad u = \cos\phi$$

$$= \frac{2\pi}{5} \left[-\frac{\cos^3\phi}{3} \right]_0^{\pi/2} = \frac{2\pi}{5} \cdot \frac{1}{3} = \underline{\frac{2\pi}{15}}$$

Anvendelser (6.4)

$A \subset \mathbb{R}^n$ begrenset Jordan-målbar

- $$\text{Volum}(A) = \int \cdots \int_A 1 \, dx_1 \cdots dx_n$$

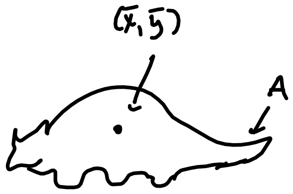
- La $f(x_1 .. x_n)$ være kontinuerlig og positiv på A.

  Dersom vi tenker på $f(x_1 .. x_n)$ som massetettheten til A så er

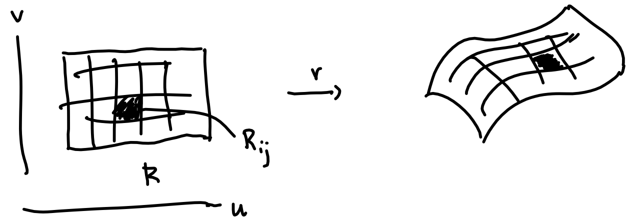
  $$\text{Masse}(A) = \int \cdots \int_A f(x_1 .. x_n) \, dx_1 .. dx_n$$

- Massemiddelpunktet til A: $(\bar{x}_1, \dots, \bar{x}_n) \in A$ er

  $$\bar{x}_i = \frac{1}{\text{Masse}(A)} \int \cdots \int x_i \cdot f(x_1 .. x_n) \, dx_1 .. dx_n$$



Arealer til parametriserte flater



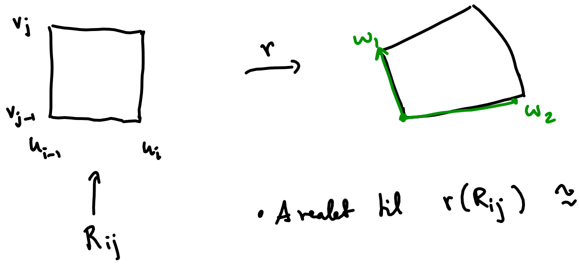
$T = r(R) \subset \mathbb{R}^3$

$$(u,v) \longmapsto r(u,v) = (x(u,v),\, y(u,v),\, z(u,v))$$

Spørsmål: Hvordan regne ut overflatearealet til T?

Π partisjon av R

$$\rightsquigarrow \text{Areal}(T) = \sum_{i,j} \text{Areal}(r(R_{ij}))$$



• Arealet til $r(R_{ij}) \approx$ arealet til parallellogrammet utspent av w_1 og w_2

$$= |w_1 \times w_2|$$

$$w_1 = r(u_{i-1}, v_j) - r(u_{i-1}, v_{j-1})$$

$$w_2 = r(u_i, v_{j-1}) - r(u_{i-1}, v_{j-1})$$

Bruker at

$$r(u_i, v_{j-1}) - r(u_{i-1}, v_{j-1}) \approx \frac{\partial r}{\partial u}(u_{i-1}, v_{j-1}) \cdot (u_i - u_{i-1})$$

$$r(u_{i-1}, v_j) - r(u_{i-1}, v_{j-1}) \approx \frac{\partial r}{\partial v}(u_{i-1}, v_{j-1})\, (v_j - v_{j-1})$$

$$\rightsquigarrow \text{Areal}(r(R)) = \sum_{i,j} \left| \frac{\partial r}{\partial u}(u_{i-1}, v_{j-1}) \times \frac{\partial r}{\partial v}(u_{i-1}, v_{j-1}) \right| (u_i - u_{i-1})(v_j - v_{j-1})$$

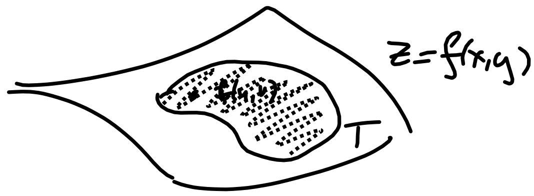
= Riemann sum for integralet

$$\iint_R \left| \frac{\partial r}{\partial u}(u,v) \times \frac{\partial r}{\partial v}(u,v) \right| du\, dv$$

Setning: La $r : \underset{[a,b]\times[c,d]}{\overset{}{R}} \longrightarrow \mathbb{R}^3$ parametrisere en flate T og anta at r har kontinuerlige partielle deriverte. Da er flatearealet til T gitt ved

$$\boxed{\iint_R \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du\, dv}$$

Spesielt tilfelle: flater på formen $z = f(x,y)$



Parametrisering:

$r(u,v) = (u,\ v,\ f(u,v))$ (*)

$$\Rightarrow \quad \frac{\partial r}{\partial u} = \left(1,\ 0,\ \frac{\partial f}{\partial u}(u,v)\right)$$

$$\frac{\partial r}{\partial v} = \left(0,\ 1,\ \frac{\partial f}{\partial v}(u,v)\right)$$

$$\Rightarrow \quad \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial u} \\ 0 & 1 & \frac{\partial f}{\partial v} \end{vmatrix} = -\frac{\partial f}{\partial u} i - \frac{\partial f}{\partial v} j + k$$

$$\Rightarrow \quad \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}$$

Setning: Arealet til flaten T parametrisert ved (*) er gitt ved

$$\iint_R \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}\ du\,dv.$$