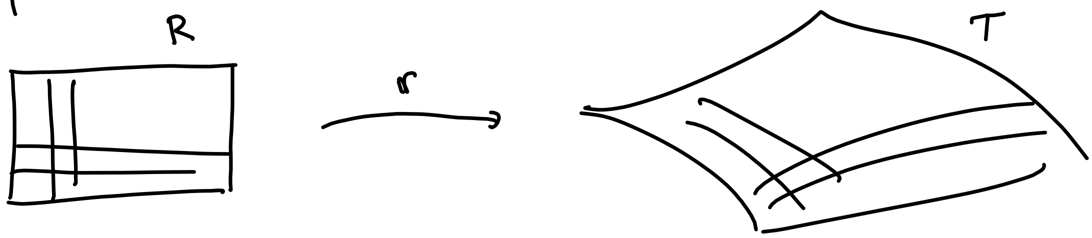
Recap

Setup:



[$\mathbf{r}$ 1-1, kontinuerlig deriverbar parametrisering av $T = \mathbf{r}(R)$]

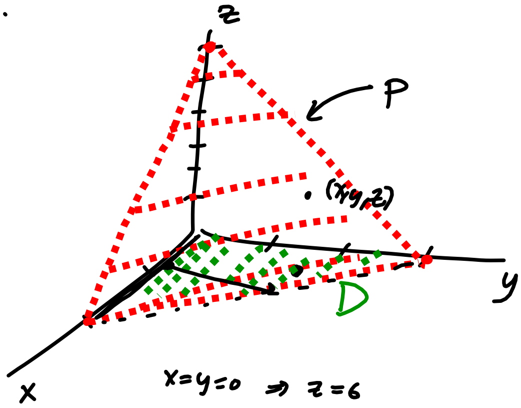
$$\text{Areal}(T) = \iint_R \left| \frac{\partial \mathbf{r}}{\partial u}(u,v) \times \frac{\partial \mathbf{r}}{\partial v}(u,v) \right| du\,dv$$

Spesialtilfelle: flater på formen $z = f(x,y)$

$\hookrightarrow \mathbf{r}(u,v) = (u, v, f(u,v))$

$$\rightsquigarrow \text{Areal}(T) = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\,dy$$

Eks Finn flatearealet av planet P $3x+2y+z=6$ som ligger i 1. kvadrant.



$\rightsquigarrow$ $z = 6-2y-3x$ $(z=f(x,y))$

Finner området D:

$z=0$ $6-2y-3x=0 \Leftrightarrow \underline{y=-\frac{3}{2}x+3}$

$x=y=0 \Rightarrow z=6$

$x=z=0 \Rightarrow y=3$

$y=z=0 \Rightarrow x=2$

$\rightsquigarrow$ D er gitt ved

$0 \le x \le 2$

$0 \le y \le -\frac{3}{2}x+3$

$\rightsquigarrow$ Får en parametrisering av P gitt ved

$$(x,\, y,\, \underbrace{6-2y-3x}_{f(x,y)})$$

$x \in [0,2]$

$y \in [0, -\frac{3}{2}x+3]$

$$\frac{\partial f}{\partial x} = -3$$

$$\frac{\partial f}{\partial y} = -2$$

$\rightsquigarrow$ Areal $= \iint_D \sqrt{1+(-3)^2+(-2)^2}\, dx\, dy$

$$= \int_0^2 \int_0^{-\frac{3}{2}x+3} \sqrt{14}\, dy\, dx$$

$$= \sqrt{14} \int_0^2 -\frac{3}{2}x+3\, dx$$

$$= \underline{3\sqrt{14}}$$

I formelen over trenger vi $\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right|$.

- Sylinderkoordinater :

  $x = r \cos\theta$

  $y = r \sin\theta \quad \rightsquigarrow \quad \left| \frac{\partial r}{\partial z} \times \frac{\partial r}{\partial r} \right| = r$

  $z = z$

- Kulekoordinater : kule av radius R

  $x = R \sin\phi \cos\theta$

  $y = R \sin\phi \sin\theta \quad \rightsquigarrow \quad \left| \frac{\partial r}{\partial \phi} \times \frac{\partial r}{\partial \theta} \right| = R^2 \sin\phi.$

  $z = R \cos\theta$

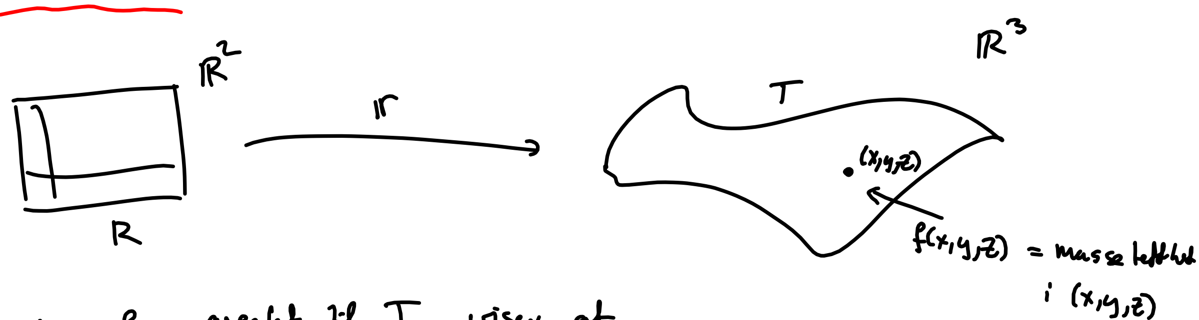
Eks Finn overflatearealet til en kule (B) med radius R.

$$\text{Areal}(B) = \int_0^{\pi}\int_0^{2\pi} \left| \frac{\partial r}{\partial \phi} \times \frac{\partial r}{\partial \theta} \right| d\theta\, d\phi = \int_0^{\pi}\int_0^{2\pi} R^2 |\sin\phi|\, d\theta\, d\phi$$

$$= 2\pi R^2 \int_0^{\pi} |\sin\phi|\, d\phi$$

$$= \underline{4\pi R^2}$$

Flateintegraler



Argumentene for arealet til T viser at

$$\text{Massen til hele flaten} = \iint_R f(\mathbf{r}(u,v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\,dv$$

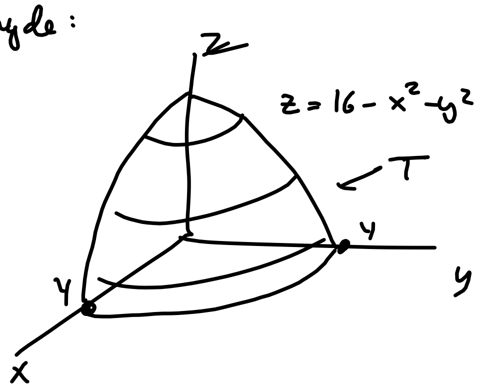
Defn Anta at T er en parametrisert flate via $\mathbf{r} : R \longrightarrow T$ og la $f : T \longrightarrow \mathbb{R}$ være en kontinuerlig funksjon.
Da kaller vi integralet

$$\iint_T f \, dS := \iint_R f(\mathbf{r}(u,v)) \cdot \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\,dv$$

for flateintegralet til f over T.

Eks La T være følgende mengde:

$$z = 16 - x^2 - y^2 \qquad x, y \geq 0$$

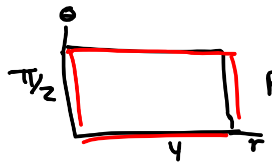


Finn $\displaystyle\iint_T xy\, dS$.

Finner først en parametrisering.

Bruker sylinderkoordinater:

$$x = r\cos\theta \qquad \theta \in [0, \pi/2]$$
$$y = r\sin\theta \qquad r \in [0, 4]$$
$$z = 16 - x^2 - y^2 = 16 - r^2$$



$$\left| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right| = \left| \begin{matrix} i & j & k \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{matrix} \right|$$

$$= \left| \left(+2r^2\cos\theta,\ -2r^2\sin\theta,\ \underbrace{r\cos^2\theta + r\sin^2\theta}_{=r} \right) \right|$$

$$= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= r\sqrt{4r^2+1}.$$

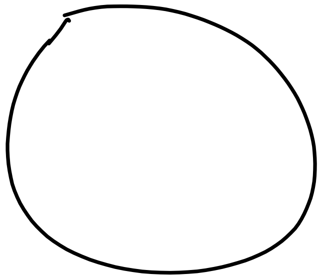
$$\rightsquigarrow \iint_T xy\, dS = \int_0^4 \int_0^{\pi/2} \overbrace{(r\cos\theta)}^{x}\overbrace{(r\sin\theta)}^{y}\, r\sqrt{4r^2+1}\, d\theta\, dr$$

$$= \int_0^4 \int_0^{\pi/2} \frac{1}{2}\sin 2\theta \cdot r^3\sqrt{4r^2+1}\, d\theta\, dr$$

$$= \frac{1}{2}\left(\int_0^4 r^3\sqrt{4r^2+1}\, dr \right)\left(\int_0^{\pi/2} \sin 2\theta\, d\theta \right)$$

$$= \ldots$$

Eks La $T = \left\{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \right\}$



T

Regn ut $\iint_T z^2 \, dS$.

Kulekoordinater: $\left| \frac{\partial r}{\partial \phi} \times \frac{\partial r}{\partial \theta} \right| = \sin \phi$

$$\iint z^2 \, dS = \int_0^{\pi} \int_0^{2\pi} (\sin \phi)(\cos \phi)^2 \, d\theta \, d\phi$$

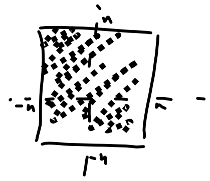
$$= 2\pi \left[-\frac{1}{3} \cos^3 \phi \right]_0^{\pi} = \underline{\frac{4\pi}{3}}$$

Uegentlige integraler

Har hittil sett på $\iint_A f \, dxdy$ der $A \subset \mathbb{R}^2$ er begrenset

Hva med ubegrensede mengder?

Notasjon $\quad K_n = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} |x| \leq n \\ |y| \leq n \end{array} \right\}$



La $A \subset \mathbb{R}^2$ være en mengde, og $f: A \to \mathbb{R}$ være en ikke negativ funksjon ($f(x) \geq 0$ for alle $x \in A$).

Vi definerer

$$\iint_A f(x,y)\, dxdy := \lim_{n \to \infty} \iint_{K_n \cap A} f(x,y)\, dxdy$$

⟵ disse er begrenset ⇝ kan integrere dersom hver $K_n \cap A$ er Jordan-målbar.

Setning 6.8.3 Vi kan også regne ut $\iint_A f \, dxdy$ som

$$\iint_A f \, dxdy = \lim_{n \to \infty} \iint_{A \cap B(0,n)} f(x,y)\, dxdy$$

der $B(0,n)$ er kula med radius n, senter i 0.

For generelle funksjoner $f: A \to \mathbb{R}$ (ikke bare ikke-negative) skriver vi

$$f_+(x) = \begin{cases} f(x) & \text{dersom } f(x) \geq 0 \\ 0 & \text{ellers} \end{cases}$$

$$f_-(x) = \begin{cases} -f(x) & \text{dersom } f(x) \leq 0 \\ 0 & \text{ellers} \end{cases}$$

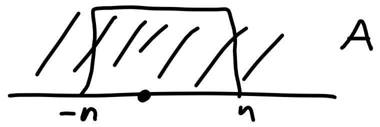
$\rightsquigarrow \quad f(x) = f_+(x) - f_-(x).$

Defn Vi sier at $\iint_A f \, dxdy$ konvergerer dersom både $\iint_A f_+(x,y)\, dxdy$ og $\iint_A f_-(x,y)\, dxdy$ eksisterer.

I så fall, definerer vi

$$\iint_A f(x,y)\, dxdy := \iint_A f_+ \, dxdy - \iint_A f_- \, dxdy$$

Hvorfor $f_\pm$?



Eks $f(x,y) = xy$

$A = \{(x,y) \mid y \geq 0\}$

$$\iint_{K_n \cap A} xy \, dxdy = \int_0^n \int_{-n}^n xy \, dx \, dy$$

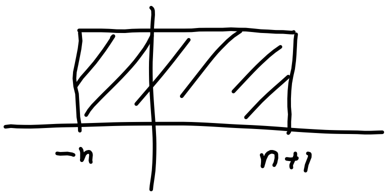
$$= \int_0^n \frac{x^2}{2} y \Big|_{-n}^{n} dy = \int_0^n 0 \, dy = 0$$

$\therefore \lim_{n\to\infty} \iint_{K_n \cap A} f \, dxdy = 0.$

På en annen side er det problematisk å si at integralet konvergerer:

- $f(x,y) = xy$ går mot ∞ når $x,y \to \infty$

- La $B_n = [-n, n+1] \times [0,n]$
  $C_n = [-n-1, n] \times [0,n]$



$$\iint_{B_n} f \, dxdy = \int_0^n \int_{-n}^{n+1} xy \, dx \, dy$$

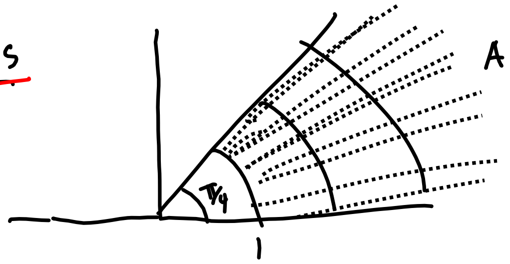
$$= \int_0^n \frac{x^2}{2} y \Big]_{-n}^{n+1} dy$$

$$= \int_0^n y \left(\frac{(n+1)^2}{2} - \frac{n^2}{2} \right) dy = \underline{\left(\frac{2n+1}{2}\right) \frac{n^2}{2}}$$

Nå ser vi at $\lim_{n\to\infty} \iint_{B_n \cap A} f \, dxdy = \infty \rightsquigarrow$ divergerer

Tilsvarende, $\lim_{n\to\infty} \iint_{C_n \cap A} f \, dxdy = -\infty$

Eks



$A =$ område s.a
$x^2+y^2 \geq 1$
og $x \geq y \geq 0$.

$$f(x,y) = \frac{1}{(x^2+y^2)^{3/2}} \quad \rightsquigarrow \quad f \geq 0 \text{ overalt } \checkmark$$

Konvergerer $\iint_A f \, dxdy$?

$$\iint_{A \cap B(0,n)} f \, dx\,dy = \int_0^{\pi/4} \int_1^n \frac{r}{(r^2)^{3/2}} \, dr\,d\theta$$

$$= \frac{\pi}{4} \int_1^n r^{-2} \, dr$$

$$= \frac{\pi}{4}\left(1 - \frac{1}{n}\right) \xrightarrow[n\to\infty]{} \underline{\frac{\pi}{4}}$$

$\rightsquigarrow$ integralet konvergerer, mot $\pi/4$.

Kan bruke dobbeltintegrasjon til å evaluere integraler i én variabel.

Eks:

Finn $I = \int_0^\infty e^{-x^2}\,dx.$

Idé: Se på $J = \iint_A e^{-x^2-y^2}\,dx\,dy$ $\quad A =$ 

$x, y \geq 0$

$$\rightsquigarrow \quad J = \lim_{n\to\infty} \int_0^n \int_0^n e^{-x^2} \underline{e^{-y^2}}\,dx\,dy$$

$$= \lim_{n\to\infty} \left(\int_0^n e^{-x^2}\,dx\right)\left(\int_0^n e^{-y^2}\,dy\right)$$

$$= I \cdot I = I^2$$

På en annen side, i polarkoordinater:

$$J = \lim_{R\to\infty} \int_0^{\pi/2} \int_0^R e^{-r^2} \cdot r\,dr\,d\theta$$

$$= \lim_{R\to\infty} \frac{\pi}{2} \int_0^R e^{-r^2} \cdot \underline{r}\,dr \qquad u = -r^2$$

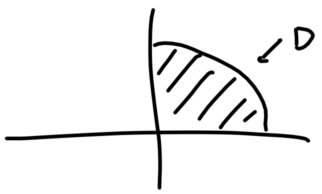
$$= \frac{\pi}{2} \cdot \lim_{R\to\infty} \left[-\frac{1}{2} e^{-r^2}\right]_0^R = \frac{\pi}{2} \cdot \frac{1}{2}$$

$$= \underline{\frac{\pi}{4}}$$

$$\rightsquigarrow \quad I = \sqrt{J} = \frac{\sqrt{\pi}}{2}$$

$$\therefore \quad \underline{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

Finn $\displaystyle\iint_D \frac{y}{\sqrt{1-x^2-y^2}}\,dx\,dy$ $\quad D = \left\{ \begin{array}{l} x \geq 0 \\ y \geq 0 \\ x^2+y^2 \leq 1 \end{array} \right\}$



Ser på

$$\lim_{R\to 1} \int_0^R \int_0^{\pi/2} \frac{r\sin\theta}{\sqrt{1-r^2}} \cdot r \, d\theta\, dr$$

$$= \lim_{R\to 1} \left(\int_0^{\pi/2} \sin\theta \, d\theta \right) \left(\int_0^R \frac{r^2}{\sqrt{1-r^2}}\, dr \right)$$

$$= \lim_{R\to 1} \int_0^R \frac{r^2}{\sqrt{1-r^2}}\, dr$$

$$= \lim_{R\to 1} \frac{1}{2} \Big[\arcsin r - r\sqrt{1-r^2} \Big]_0^R$$

$$= \lim_{R\to 1} \frac{1}{2} \Big[\arcsin R \Big] < \infty$$

$\rightsquigarrow$ integralet konvergerer.