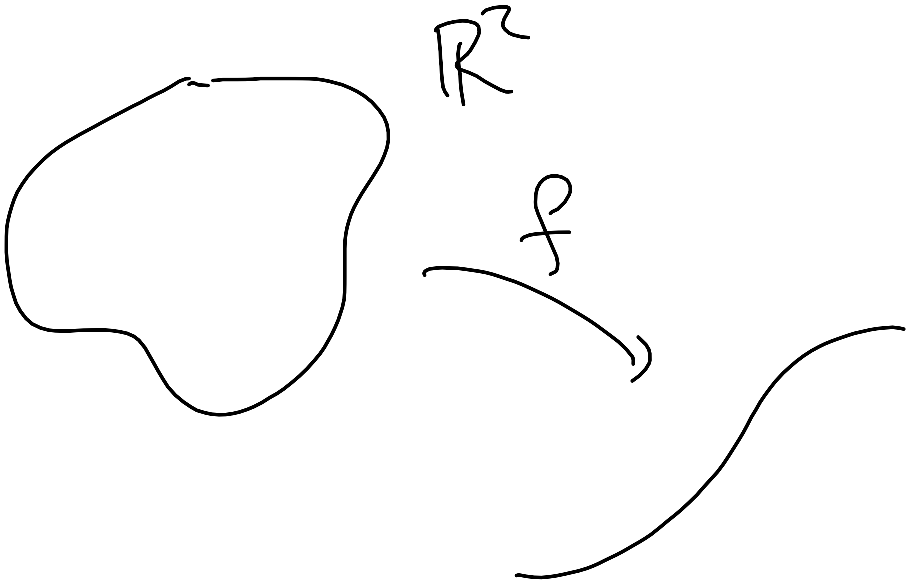
Oppgaver

6.1 1, 2a

6.2 1, 2a, 3

6.3 1, 2a, 3



$$\iint_R f(x,y)\,dx\,dy$$

$R = \square$

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dx\,dy$$

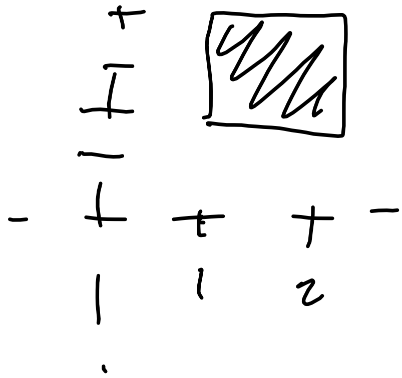
$$= \int_a^b \left(\int_c^d f(x,y)\,dx \right) dy$$

$R =$ 

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left(\int_{\psi(x)}^{\varphi(x)} f(x,y)\,dy \right) dx$$

6.1.1

a) $\iint xy \, dx\,dy \qquad R = [1,2]\times[2,4]$



$$= \int_1^2 \int_2^4 xy \, dx\,dy$$

$$= \int_1^2 \left(\int_2^4 xy \, dy \right) dx$$

$$= \int_1^2 x \frac{y^2}{2} \Big|_2^4 dx = \int_1^2 x\left(\frac{4^2}{2} - \frac{2^2}{2}\right) dx$$

$$= 6 \int_1^2 x \, dx = 6 \frac{x^2}{2} \Big|_1^2$$

$$= 6\left(\frac{2^2}{2} - \frac{1}{2}\right) = 3(4-1)$$

$$= \underline{9}$$

c) $\int_{-1}^{1}\int_{0}^{1} x^2 e^y \, dx\,dy$

$= \int_{-1}^{1}\left(x^2 \int_{0}^{1} e^y \, dy\right) dx$

$= \int_{-1}^{1} x^2 \left. e^y \right|_0^1 dx$

$= \int_{-1}^{1} x^2 (e-1)\, dx$

$= (e-1)\int_{-1}^{1} x^2 \, dx = (e-1) \left. \frac{x^3}{3} \right|_{-1}^{1}$

$= \frac{(e-1)}{3}(1+1) = \frac{2(e-1)}{3}$

d) $\int_1^2 \int_\pi^{2\pi} x \cos(xy)\, dx\, dy \quad R = [1,2] \times [\pi, 2\pi]$

$\int_1^2 \left(\int_\pi^{2\pi} x \cos(xy)\, dy \right) dx \qquad u = xy$

$du = x\, dy$

$dy = \frac{du}{x}$

$= \int_1^2 \left(\int_{\pi x}^{2\pi x} \cancel{x} \cos u \frac{du}{\cancel{x}} \right) dx$

$= \int_1^2 \left(\int_{\pi x}^{2\pi x} \cos u\, du \right) dx$

$= \int_1^2 - \sin u \Big|_{\pi x}^{2\pi x} dx$

$= \int_1^2 - \sin(2\pi x) + \sin(\pi x)\, dx$

$= - \int_1^2 \sin(2\pi x)\, dx + \int_1^2 \sin(\pi x)\, dx$

$= -\frac{1}{2\pi} \cos(2\pi x) \Big|_1^2 + \frac{1}{\pi} \cos(\pi x) \Big|_1^2$

$= \underbrace{-\frac{1}{2\pi} \left(\cos(4\pi) - \cos(2\pi) \right)}_{=0} + \frac{1}{\pi} \left(\cos 2\pi - \cos \pi \right)$

$= \frac{2}{\pi}$

e) $\int_0^2 \int_1^2 x\, y\, e^{x^2 y}\, dy\, dx = I$

$R = [0,2] \times [1,2]$

$\iint xy\, e^{x^2 y}\, dy\, dx$

$u = x^2 y$

$du = 2xy\, dx$

$= \iint xy\, e^u \frac{du\, dx}{2xy}$

$dx = \frac{du}{2xy}$

$= \frac{1}{2} \iint e^u\, du\, dx$

$I = \int_1^2 \frac{1}{2} e^u \Big|_0^{4y} dy = \frac{1}{2} \int_1^2 e^{4y} - 1\, dy$

$= \frac{1}{2} \left(\frac{1}{4} e^{4y} \Big|_1^2 - 1 \right)$

$= \frac{1}{2} \left(\frac{1}{4} e^8 - \frac{1}{4} e^4 - 1 \right)$

$= \frac{1}{8} \left(e^8 - e^4 - 4 \right)$

f) $\displaystyle\int_1^e \int_1^e \underline{\ln(xy)}\, dx\,dy$

$$\int_1^e \int_1^e \ln x + \ln y \; dx\,dy$$

$$= \int_1^e \int_1^e \ln x \; dx\,dy + \int_1^e \int_1^e \ln y \; dx\,dy$$

$$= 2 \int_1^e \int_1^e \ln x \, dx\,dy$$

$$= 2 \int_1^e \ln x \, dx \int_1^e dy$$

$$= 2(e-1) \int_1^e \ln x \, dx$$

$\int \ln x \, dx$

$= x \ln x - x$

$$= 2(e-1) \left(x \ln x - x \right) \Big|_1^e$$

$$= 2(e-1)\left(e\cdot\cancel{1} - \cancel{e} - (0-1) \right)$$

$$= \underline{2(e-1)}\,.$$

g) $\iint_R \frac{1}{1+x^2y}\,dx\,dy$ $R=[1,\sqrt{3}]\times[0,1]$

$= \int_1^{\sqrt{3}} \left(\int_0^1 \frac{dy}{1+x^2y} \right) dx$ $u = 1+x^2y$

$= \int_1^{\sqrt{3}} \int_1^{1+x^2} \frac{1}{u} \cdot \frac{du}{x^2} \Big) dx$ $du = x^2 dy$

$dy = \frac{du}{x^2}$

$= \int_1^{\sqrt{3}} \frac{\log(1+x^2)}{x^2}\,dx$ διολοκλήρωση κατά παράγοντες:

$\int \frac{\log(1+x^2)}{x^2} dx =$ $f = \log(1+x^2)$

$= \log(1+x^2)\left(-\frac{1}{x}\right)$

$- \int \frac{\cancel{2x}}{1+x^2}\left(-\frac{1}{\cancel{x}}\right) dx$ $df = \frac{2x}{1+x^2}$

$g = -\frac{1}{x}$ $dg = \frac{dx}{x^2}$

$\int f\,dg = fg - \int g\,df$

$= \log(1+x^2)\left(-\frac{1}{x}\right) + 2\int \frac{dx}{1+x^2}$

$= -\frac{1}{x}\log(1+x^2) + 2\arctan x$

$\int_1^{\sqrt{3}} \cdots dx = 2\arctan x - \frac{1}{x}\log(1+x^2) \Big|_1^{\sqrt{3}}$

$= \left(2\pi/3 - \frac{1}{\sqrt{3}}\log(4)\right)$

$- \left(2\pi/4 - \log 2\right)$ $2\pi/3 - \pi/2 = \pi/6$

$= \frac{\pi}{6} + \log 2 - \frac{2}{\sqrt{3}}\log 2$

$= \pi/6 + \log 2\left(1 - \frac{2\sqrt{3}}{3}\right)$

6.2

a) $\iint_R x^2 y \, dx\,dy$

R: $0 \le x \le 2$
$0 \le y \le x$

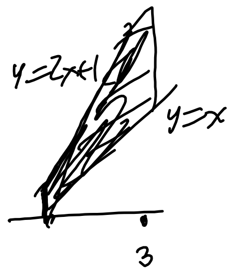
$= \int_0^2 \left(\int_0^x x^2 y \, dy \right) dx$

$= \int_0^2 x^2 \frac{x^2}{2} \, dx = \int_0^2 \frac{x^4}{2} \, dx$

$= \frac{1}{2} \frac{x^5}{5} \Big|_0^2 = \frac{16}{5}$

b) $\iint_R (x + 2xy) \, dy\,dx$

R: $0 \le x \le 3$
$x \le y \le 2x+1$



$= \int_0^3 \int_x^{2x+1} (x+2xy) \, dy \, dx$

$= \int_0^3 xy + xy^2 \Big|_x^{2x+1} dx$

$= \int_0^3 x(2x+1) + x(2x+1) - (x^2 + x x^2) \, dx$

$= \int_0^3 3x^3 + 5x^2 + 2x \, dx$

$= \frac{3}{4} x^4 + \frac{5}{3} x^3 + x^2 \Big|_0^3$

$= \frac{243}{4} + \frac{27.5}{3} + 3^2$

$= \frac{459}{4}$

c) $\iint_R y \, dx\,dy \qquad R: \; 1 \le y \le 2, \quad y \le x \le y^2$

$$= \int_1^2 \left(\int_y^{y^2} y \, dx \right) dy$$

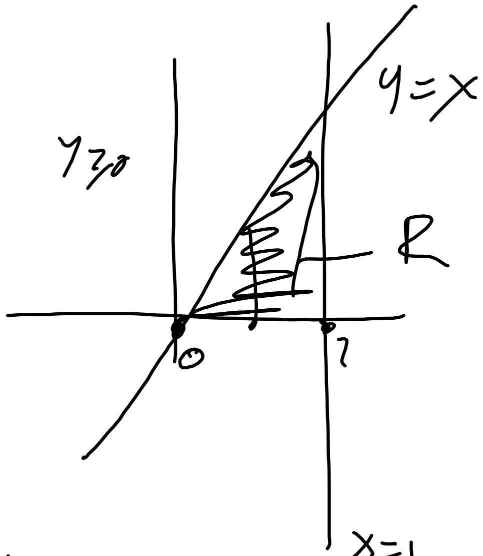
$$= \int_1^2 y(y^2 - y)\, dy = \int_1^2 y^3 - y^2 \, dy$$

$$= \frac{y^4}{4} - \frac{y^3}{3} \Big|_1^2$$

$$= \left(\frac{16}{4} - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)$$

$$= \frac{15}{4} - \frac{7}{3} = \underline{\frac{17}{12}}$$

6.2.1 e)



$$\iint_R e^{x^2}\,dx\,dy$$

$$= \int_0^1 \left(\int_0^x \underline{e^{x^2}}\, dy \right) dx$$

$$= \int_0^1 x e^{x^2}\, dx = \frac{1}{2}\int_0^1 e^u\, du$$

$u = x^2$
$du = 2x\,dx$
$dx = \dfrac{du}{2x}$

$$= \frac{1}{2} e^u \Big|_0^1 = \frac{1}{2}(e-1)$$

h)

$$\iint_R \frac{1}{\sqrt{1-y^2}}\,dx\,dy \qquad R: \; 0 \leq y \leq \sin x, \quad 0 \leq x \leq \frac{\pi}{2}$$

$$= \int_0^{\pi/2} \left(\int_0^{\sin x} \frac{1}{\sqrt{1-y^2}}\, dy \right) dx$$

$$= \int_0^{\pi/2} \arcsin y \Big|_0^{\sin x} dx$$

$$= \int_0^{\pi/2} x\, dx = \left(\frac{\pi}{2}\right)^2 \cdot \frac{1}{2} = \frac{\pi^2}{8}$$

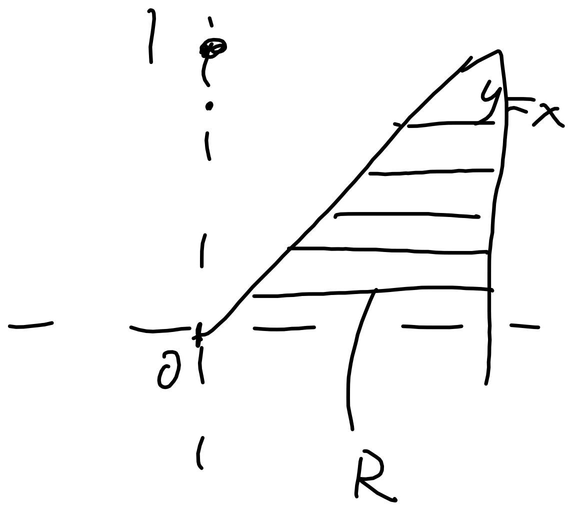
3) $\int_0^1 \left(\int_y^1 e^{x^2}\, dx \right) dy = \iint_R e^{x^2}\, dx\, dy$

R : $0 \le y \le 1$

$y \le x \le 1$

$\Leftrightarrow$ $0 \le x \le 1$

$0 \le y \quad x$



$$I = \iint_R e^{x^2}\, dx\, dy = \int_0^1 \left(\int_0^x e^{x^2}\, dy \right) dx$$

$$= \int_0^1 x e^{x^2}\, dx = \frac{1}{2} e^{x^2} \Big|_0^1 = \frac{1}{2}(e-1)$$

$u = x^2$

b) $$\int_0^{\pi/2}\left(\int_x^{\pi/2} \frac{\sin y}{y}\,dy\right)dx = \iint_R$$

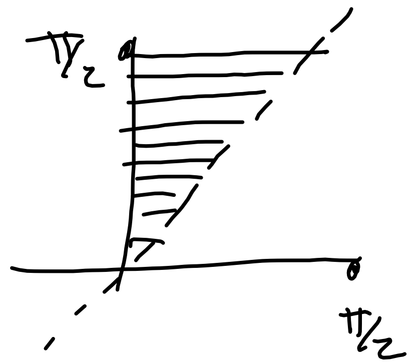
R: $0 \le x \le \pi/2$

$x \le y \le \pi/2$

$\Updownarrow$

R: $0 \le y \le \pi/2$

$0 \le x \le y$



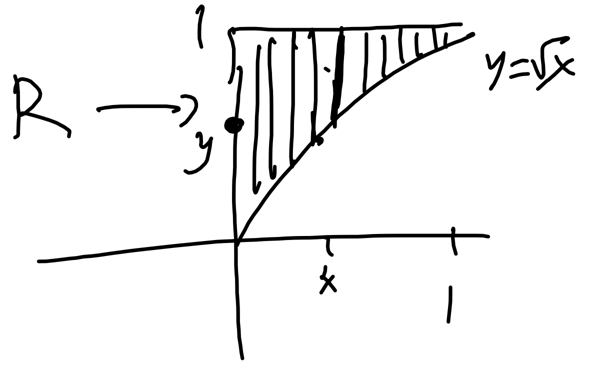
$$\Rightarrow \iint_R \frac{\sin y}{y}\,dx\,dy = \int_0^{\pi/2}\left(\int_0^{y} \frac{\sin y}{y}\,dx\right)dy$$

$$= \int_0^{\pi/2} \frac{\sin y}{\cancel{y}} \cdot \cancel{y}\,dy$$

$$= \int_0^{\pi/2} \sin y\,dy = -\cos y\Big|_0^{\pi/2}$$

$$= \underline{1}$$

c) $\displaystyle\int_0^1 \left(\int_{\sqrt{x}}^1 e^{x/y^2}\, dy \right) dx = \iint_R \cdots$



$R: \quad 0 \le x \le 1$

$\quad\;\; \sqrt{x} \le y \le 1$

$\Updownarrow$

$R; \quad 0 \le y \le 1$

$\quad\;\; 0 \le x \le y^2$

$$\iint_R e^{x/y^2}\, dx\,dy = \int_0^1 \left(\int_0^{y^2} e^{x/y^2}\, dx \right) dy$$

$u = x/y^2$

$du = dx/y^2$

$dx = y^2 du$

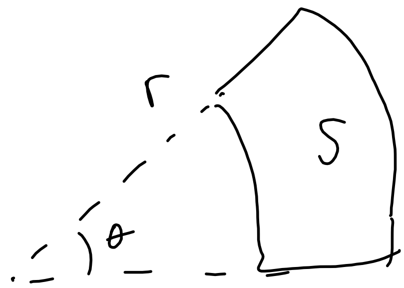
$$= \int_0^1 \left(y^2 \cdot e^{x/y^2} \Big|_0^{y^2} \right) dy$$

$$= \int_0^1 y^2 e^{\cancel{y^2/y^2}} - y^2 \, dy$$

$$= \int_0^1 y^2 (e-1)\, dy$$

$$= (e-1) \cdot \int_0^1 y^2 dy = \underline{\underline{\frac{e-1}{3}}}$$

6.3 Polarkoordinater



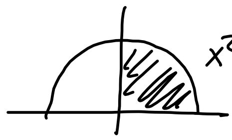
$$\iint_S f(x,y)\,dx\,dy$$

$$= \iint_R f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

$$R\begin{cases} a \le r \le b \\ \alpha \le \theta \le \beta \end{cases}$$

$$r^2 = x^2 + y^2$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

1a) $\displaystyle\iint_R x\,y^2\,dx\,dy$ R:  $x^2+y^2 \le 9$

$$= \int_0^{\pi/2}\int_0^3 (r\cos\theta)(r\sin\theta)^2\, r\,dr\,d\theta \qquad x = r\cos\theta,\quad y = r\sin\theta$$

$$= \int_0^3\int_0^{\pi/2} r^4 \cos\theta \sin^2\theta\,d\theta\,dr$$

$$= \left(\int_0^3 r^4\,dr\right)\left(\int_0^{\pi/2} \cos\theta\sin^2\theta\,d\theta\right) \qquad u = \sin\theta,\quad du = \cos\theta\,d\theta$$

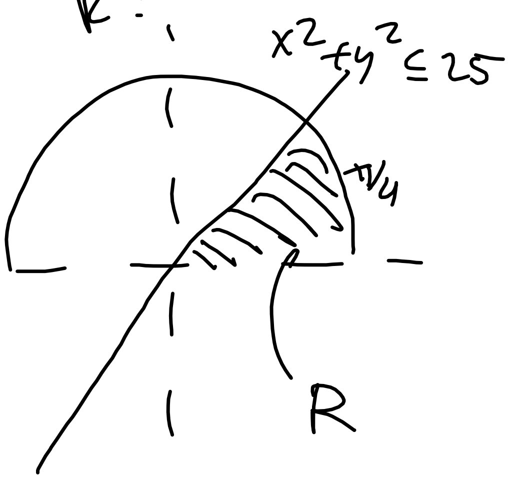
$$\int_0^3 r^4\,dr = \frac{1}{5}r^5\Big|_0^3 = \frac{243}{5}$$

$$\int_0^{\pi/2} \cos\theta\sin^2\theta\,d\theta = \int_0^1 u^2\,du = \frac{1}{3}$$

$$\therefore \iint_R = \frac{243}{5}\cdot\frac{1}{3} = \underline{\frac{81}{5}}$$

b) $\iint_R (x^2+y^2)\,dx\,dy$

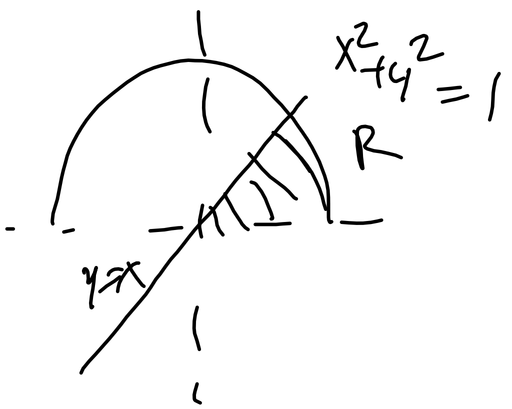
R :



$= \int_0^{\pi/4} \int_0^5 r^2 \cdot r \, dr \, d\theta$

$= \int_0^{\pi/4} \frac{r^4}{4} \Big|_0^5 \, d\theta = \frac{\pi}{4} \cdot \frac{5^4}{4} = \frac{625\pi}{16}$

d) $\iint_R xy \, dx\,dy$



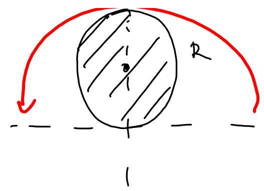
$= \int_0^1 \int_0^{\pi/4} r\cos\theta \, r\sin\theta \, r \, dr\, d\theta$

$= \left(\int_0^1 r^3 \, dr\right)\left(\int_0^{\pi/4} \cos\theta \sin\theta \, d\theta\right)$

$= \frac{1}{4}\left(\int_0^{\pi/4} \frac{\sin 2\theta}{2} \, d\theta\right)$ $\quad \sin 2\theta = 2\cos\theta\sin\theta$

$= \frac{1}{4} \cdot \frac{\cos 2\theta}{4} \Big|_0^{\pi/4} = \frac{1}{16}(0+1) = \frac{1}{16}$

3 R:



a) Vis at dersom f er kontinuerlig på R

så er

$$\iint_R f\,dx\,dy = \int_0^{\pi}\int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

b) Regn ut $\iint_R \sqrt{x^2+y^2}\,dx\,dy$

---

a): R er gitt ved $x^2+(y-1)^2 \leq 1$

$$\Leftrightarrow x^2+y^2-2y+1 \leq 1 \qquad x = r\cos\theta$$

$$\Leftrightarrow r^2 - 2r\sin\theta \leq 0 \qquad y = r\sin\theta$$

$$\Leftrightarrow r \leq 2\sin\theta \qquad (r \neq 0)$$

$$\therefore \iint_R f\,dx\,dy = \int_0^{\pi}\int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

b) $\iint \sqrt{x^2+y^2}\,dx\,dy$

$$= \int_0^{\pi}\int_0^{2\sin\theta} r\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2\sin\theta} r^2\,dr\,d\theta$$

$$= \int_0^{\pi} \left.\frac{r^3}{3}\right|_0^{2\sin\theta} d\theta$$

$$= \frac{8}{3}\int_0^{\pi} \sin^3\theta\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi} \sin\theta(1-\cos^2\theta)\,d\theta \qquad \sin^3\theta = \sin\theta(1-\cos^2\theta)$$

$$= \frac{8}{3}\int_1^{-1} (1-u^2)\cdot(-1)\,du \qquad u = \cos\theta$$

$$= \frac{8}{3}\int_{-1}^{1} 1-u^2\,du \qquad du = -\sin\theta\,d\theta$$

$$= 2\cdot\frac{8}{3}\int_0^1 1-u^2\,du$$

$$= \frac{16}{3}\left[u - \frac{u^3}{3}\right]_0^1 = \frac{16}{3}\left[1-\frac{1}{3}\right]$$

$$= \frac{16}{3}\cdot\frac{2}{3} = \underline{\underline{\frac{32}{9}}}$$