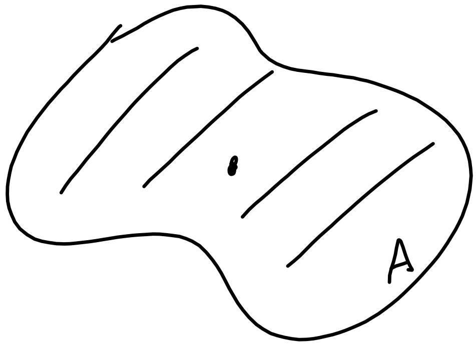
Oppgaver

		Ønsker:
6.4	1 adf, 2, 5	5
6.8	1, 2, 3	3
6.9	1, 2abd	2d
6.10	1, 2ab	1c, 2b
6.11	1, 3, 6	3

6.4



- $\text{Areal}(A) = \iint_A 1 \, dx\,dy$

- Massemiddelpunkt $= (\bar{x}, \bar{y})$ $\quad f : A \to \mathbb{R}$

$$\bar{x} = \frac{\iint_A x\, f(x,y)\, dx\,dy}{\iint_A f(x,y)\, dx\,dy} \qquad \bar{y} = \frac{\iint_A y\, f(x,y)\, dx\,dy}{\iint_A f(x,y)\, dx\,dy}$$

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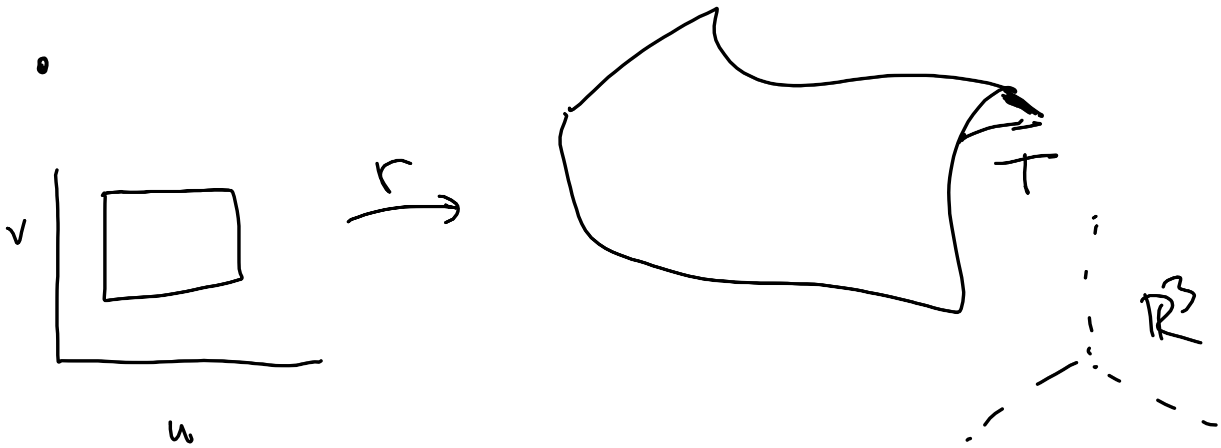


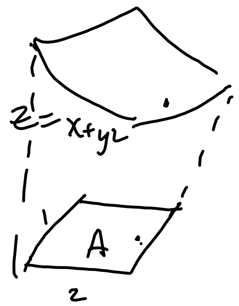
overflatearealet $:= \iint_T |r_u \times r_v| \, du\,dv$

$$\iint_T f \, dS := \iint_A f(r(u,v)) \cdot |r_u \times r_v| \, du\,dv$$

6.4.1

$$E = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 1 \\ 0 \le z \le x+y^2 \end{array} \right\}$$

$$\text{Volum}(E) = \iint_A x+y^2 \, dA$$



$$= \int_0^2 \int_0^1 x+y^2 \, dy \, dx$$

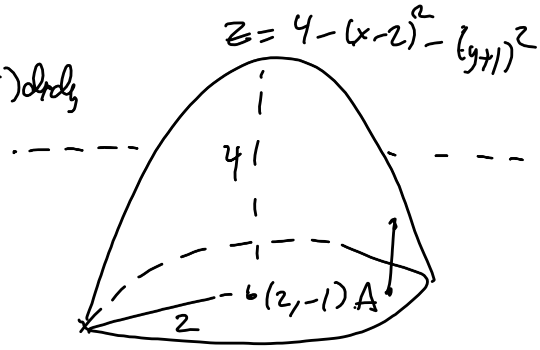
$$= \int_0^2 xy + \frac{y^3}{3} \Big|_0^1 \, dx$$

$$= \int_0^2 x + \frac{1}{3} \, dx = \frac{x^2}{2} + \frac{1}{3}x \Big|_0^2$$

$$= 2 + \frac{1}{3} = \underline{\frac{8}{3}}$$

f) E = området over xy-planet og under grafen til $4-(x-2)^2-(y+1)^2$

$$\text{Vol}(E) = \iint_A \left(4-(x-2)^2-(y+1)^2\right) dx\,dy$$



$u = x-2$
$v = x+1$ $\rightsquigarrow J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \iint_{A'} 4-u^2-v^2 \, dA$$

A' = sirkel med senter i $(0,0)$ og radius $= 2$.

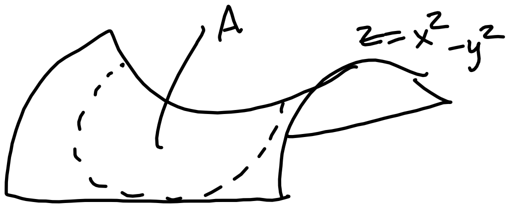
$$= \int_0^2 \int_0^{2\pi} (4-r^2)\, r \, dr \, d\theta$$

$$= \int_0^2 (4r - r^3) \underline{\int_0^{2\pi} 1 \, d\theta} \, dr$$

$$= 2\pi \left(2r^2 - \frac{1}{4}r^4 \Big|_0^2 \right) = 2\pi(8-4) = \underline{8\pi}$$

5 Finn arealet til flaten $\begin{cases} z = x^2 - y^2 \\ x^2 + y^2 \leq 4 \end{cases}$

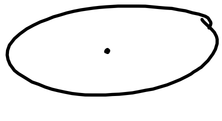
Arealet $= \iint |r_u \times r_v| \, du\, dv$



$r(u,v) = (u, v, u^2 - v^2)$

$r_u = (1, 0, 2u)$

$r_v = (0, 1, -2v)$



$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix}$$

$$= (-2u, 2v, 1) \quad \therefore \; |r_u \times r_v| = \sqrt{4u^2 + 4v^2 + 1}$$

$$\iint_A \sqrt{1 + 4u^2 + 4v^2} \, dA = \int_0^2 \int_0^{2\pi} r\sqrt{1+4r^2} \, dr \, d\theta$$

A: 

$$= 2\pi \cdot \int_0^2 r\sqrt{1+4r^2} \, dr$$

$u = 1 + 4r^2$

$du = 8r \, dr$

$dr = \frac{du}{8r}$

$$= 2\pi \int_1^{17} \cancel{r}\sqrt{u} \, \frac{du}{8\cancel{r}}$$

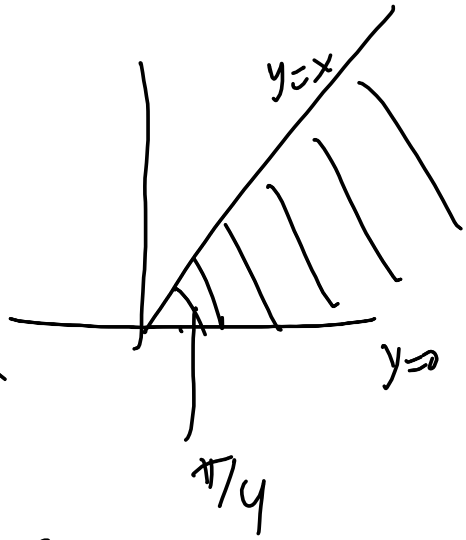
$$= \frac{\pi}{4} \int_1^{17} \sqrt{u} \, du$$

$$= \frac{\pi}{4} \, \frac{2}{3} u^{3/2} \Big|_1^{17}$$

$$= \underline{\frac{\pi}{6} \left(17^{3/2} - 1\right)}$$

6.8

1. $$\iint_A e^{-x^2-y^2}\,dx\,dy$$



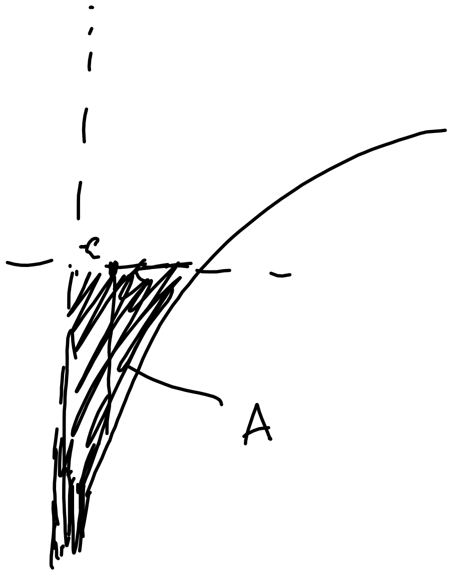
$$= \lim_{R\to\infty} \int_0^R \int_0^{\pi/4} e^{-r^2}\, r \, d\theta\, dr$$

$$= \frac{\pi}{4} \cdot \lim_{R\to\infty} \int_0^R e^{-r^2}\, r\, dr \qquad u = -r^2, \quad du = -2r\,dr$$

$$= \frac{\pi}{4} \cdot \lim_{R\to\infty} \int_0^{-R^2} \cancel{r}\, e^u \frac{du}{-2\cancel{r}} = \frac{\pi}{8} \int_{-\infty}^{0} e^u\, du$$

$$= \frac{\pi}{8} \left[e^u \right]_{-\infty}^{0} = \frac{\pi}{8}(1-0) = \underline{\frac{\pi}{8}}$$

3. Avgjør om integralet $\iint_A x\,dx\,dy$ når A er følgende område:



Dette konvergerer hvis og bare hviss

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\int_{\ln x}^{0} x\,dy\,dx < \infty$$

$$\int_{\varepsilon}^{1}\int_{\ln x}^{0} x\,dy\,dx$$

$$= \int_{\varepsilon}^{1} x\left(y\Big|_{\ln x}^{0}\right)dx = \int_{\varepsilon}^{1} x(0-\ln x)\,dx$$

$$= -\int_{\varepsilon}^{1} x\ln x\,dx = \underline{-\frac{x^2}{2}\ln x\Big|_{\varepsilon}^{1}} + \underbrace{\int_{\varepsilon}^{1}\frac{dx}{x}\cdot\frac{x^2}{2}}_{\int_{\varepsilon}^{1}\frac{x}{2}\,dx}$$

delvis integrasjon
$f = \ln x \quad df = \frac{dx}{x}$
$dg = x \quad g = \frac{x^2}{2}$

$$\lim_{\varepsilon\to 0} -x^2\ln x\Big|_{\varepsilon}^{1} = \lim_{\varepsilon\to 0}\varepsilon^2\ln\varepsilon = \lim_{\varepsilon\to 0}\frac{\ln\varepsilon}{\frac{1}{\varepsilon^2}} \qquad \frac{\infty}{\infty}$$

$$= \lim_{\varepsilon\to 0}\frac{\frac{1}{\varepsilon}}{-2\varepsilon^{-3}} = \lim_{\varepsilon\to 0} -\frac{1}{2}\varepsilon^2 = \underline{0}$$

∴ integralet konvergerer:

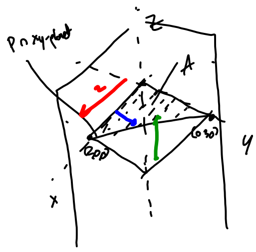
$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{x\,dx}{2} = \int_{0}^{1}\frac{x}{2}\,dx = \underline{\frac{1}{4}}$$

6.9

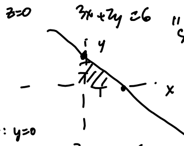
6.9.2 d) $\iiint_A 3y^2 - 3z \, dx\,dy\,dz$

A er området avgrenset av koordinatplanene og planet

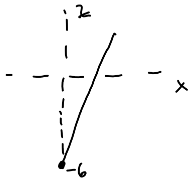
$P: \; 3x + 2y - z = 6$



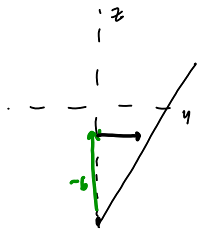
xy-planet: $z=0$ $\quad 3x+2y=6$ "sett ovenifra"



xz-planet: $y=0$ $\quad 3x - z = 6$ $\quad z = 3x-6$



yz-planet: $z = 2y - 6$



A: $x = 0 \ldots 2$

$y = 0 \ldots 3 - \frac{3}{2}x \qquad (3x + 2y = 6)$

$z = 3x + 2y - 6 \ldots 0$

$\Updownarrow$

$z = -6 \ldots 0$

$y = 0 \ldots z/2 + 3 \qquad (z = 2y - 6)$

$x = 0 \ldots \frac{1}{3}(6 + z - 2y)$

$$\iiint_A 3y^2 - 3z \, dx\,dy\,dz = \int_{-6}^{0} \int_0^{z/2+3} \int_0^{1/3(6+z-2y)} 3y^2 - 3z \, dx\,dy\,dz$$

$$= \int_{-6}^{0} \int_0^{z/2+3} 1/3\,(6+z-2y)(3y^2 - 3z)\, dy\,dz$$

$$= \int_{-6}^{0} \int_0^{z/2+3} (6+z-2y)(y^2 - z)\, dy\,dz$$

$$= \int_{-6}^{0} \int_0^{z/2+3} 6y^2 - zy^2 - 2y^3 - 6z + z^2 + 2zy \, dy\,dz$$

$$= \int_{-6}^{0} 6\frac{y^3}{3} - z\frac{y^3}{3} - \frac{2y^4}{4} - 6yz + z^2 y + \frac{2zy^2}{2} \Big|_0^{z/2+3}$$

$$= \int_{-6}^{0} \frac{27}{2} - \frac{3z^2}{4} + \frac{z^4}{96} \, dz$$

$$= \frac{27}{2}z - \frac{z^3}{4} + \frac{z^5}{480} \Big|_{-6}^{0} = \underline{\frac{216}{5}}$$

6.16

1c

Bruk sylinderkoordinater:

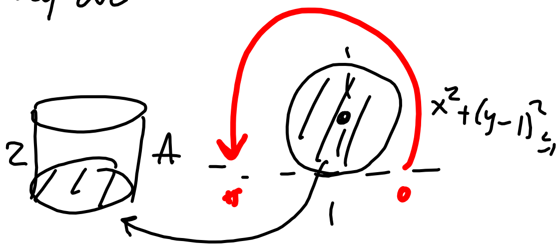
A: $x^2+(y-1)^2 \le 1$

$0 \le z \le 2$

$$\iiint_A z\sqrt{x^2+y^2}\,dx\,dy\,dz$$

$x = r\cos\theta$

$y = r\sin\theta$



$x^2+(y-1)^2 \le 1 \Leftrightarrow x^2+y^2-2y+1 \le 1$

$$\Leftrightarrow r^2 - 2r\sin\theta \le 0 \overset{r\neq 0}{\Leftrightarrow} \underline{r \le 2\sin\theta}$$

$$\iiint_A f\,dx\,dy\,dz = \int_0^2 \int_0^\pi \int_0^{2\sin\theta} zr\cdot r\,dr\,d\theta\,dz$$

$$= \left(\int_0^2 z\,dz\right)\left(\int_0^\pi \frac{r^3}{3}\Big|_0^{2\sin\theta} dr\right)$$

$$= \left(\frac{z^2}{2}\Big|_0^2\right)\left(\int_0^\pi \frac{8}{3}\sin^3\theta\,dr\right)$$

$$= 2\cdot\frac{8}{3}\int_0^\pi \sin^3\theta\,d\theta$$

$\sin^2\theta = 1-\cos^2\theta$

$$= \frac{16}{3}\int_1^{-1} -(1-u^2)\,du$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= \frac{16}{3}\int_{-1}^1 1-u^2\,du$$

$\sin^3\theta\,d\theta = \sin^2\theta\cdot\sin\theta\,d\theta$

$= \sin^2\theta\cdot(-du)$

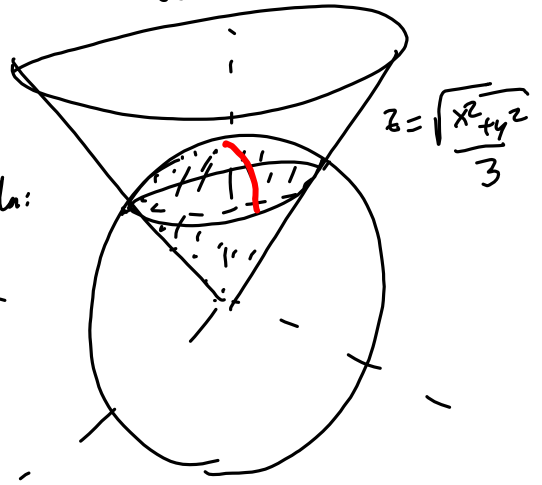
$= (1-u^2)\,du$

$$= \frac{16}{3}\left(u-\frac{u^3}{3}\right)\Big|_{-1}^1$$

$$= \frac{16}{3}\left(2-\frac{2}{3}\right) = \frac{16}{3}\cdot\frac{4}{3} = \underline{\underline{\frac{64}{9}}}$$

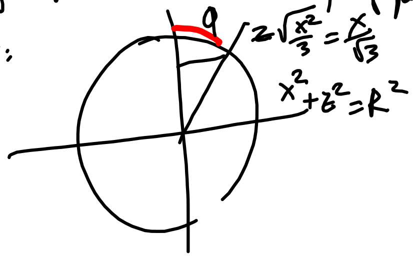
6.11.3 Finn volumet av den delen av kulen $x^2+y^2+z^2=R^2$ som ligger over

$$z=\sqrt{\frac{x^2+y^2}{3}}$$



Finn skjæringen mellom kula og kjegla:

xz-planet:



$\frac{x}{z}=\tan\phi=\sqrt{3}$ $\Rightarrow$ $\phi=\arctan(\sqrt{3})=\pi/3$

$$\int_0^R\int_0^{\pi/6}\int_0^{2\pi}\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$$

$$=\left(\int_0^R\rho^2\,d\rho\right)\cdot\left(\int_0^{\pi/3}\sin\phi\,d\phi\right)\cdot\left(\int_0^{2\pi}d\theta\right)$$

$$=\frac{\rho^3}{3}\Big|_0^R\cdot\frac{1}{2}\cdot 2\pi$$

$$=\frac{R^3}{3}\cdot 2\pi\cdot\frac{1}{2}=\underline{\frac{\pi R^3}{3}}$$

6.10.2c

Bruk kulekoordinater til å beregne $I = \iiint_A x \, dx\,dy\,dz$

$$A: \quad \begin{aligned} &x, y \geq 0 \Leftrightarrow 0 \leq \theta \leq \frac{\pi}{2} \\ &z \geq 1/2 \Leftrightarrow \rho\cos\phi \geq \frac{1}{2} \\ &x^2+y^2+z^2 \leq 1 \Leftrightarrow \rho \leq 1 \end{aligned}$$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho\cos\phi \quad \rightsquigarrow \quad z \in \left[\frac{1}{2\cos\theta}, 1\right]$$

$$I = \int_0^{\pi/2}\int_0^{\pi/3}\int_{\frac{1}{2\cos\phi}}^{1} \rho \sin\phi \cos\theta \, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \underbrace{\left(\int_0^{\pi/2} \cos\theta \, d\theta\right)}_{=1} \left(\int_0^{\pi/3} \left(\int_{\frac{1}{2\cos\phi}}^{1} \rho^3 \, d\rho\right) \sin^2\phi \, d\phi\right)$$

$$= \int_0^{\pi/3} \frac{1}{4}\rho^4 \Big|_{\frac{1}{2\cos\phi}}^{1} \sin^2\phi \, d\phi \qquad \sin^2\phi = 1-\cos^2\phi$$

$$= \frac{1}{4}\int_0^{\pi/3} 1 - \cos^2\phi - \frac{1}{16\cos^4(\phi)} + \frac{1}{16\cos^2\phi} \, d\phi$$

$$= \frac{1}{4}\int_0^{\pi/3} 1 \, d\theta - \frac{1}{4}\int_0^{\pi/3} \cos^2\phi \, d\phi - \frac{1}{16\cdot 4}\int_0^{\pi/3} \frac{1}{\cos^4\phi} \, d\phi + \frac{1}{16}\int \frac{d\phi}{\cos^2\phi}$$

- $\displaystyle\int \frac{dt}{\cos^m t} = \frac{m-1}{m-1}\int \frac{dt}{\cos^{m-2} t} + \frac{\sin t}{m-1}\frac{1}{\cos^{m-1} t}$
- $\displaystyle\int \frac{dt}{\cos^2 t} = \tan t$
- $\displaystyle\int \frac{dt}{\cos^4 t} = \frac{\sin t}{3}\frac{1}{\cos^3 t} + \frac{2}{3}\int \frac{dt}{\cos^2 t} = \frac{\sin t}{3}\frac{1}{\cos^3 t} + \frac{2}{3}\tan t$
- $\displaystyle\int \cos^2 t \, dt = \frac{1}{2}x + \frac{1}{2}\sin x \cos x$

$$\int_0^{\pi/3} \cos^2 t \, dt = \pi/6 + \frac{1}{4} \sin \pi/3 = \pi/6 + \frac{\sqrt{3}}{8}$$

$$\int_0^{\pi/3} \cos^{-2} t \, dt = \tan t \Big|_0^{\pi/3} = \sqrt{3}$$

$$\int_0^{\pi/3} \cos^{-4} t \, dt = \frac{1}{3} \tan t \cdot \frac{1}{\cos^2 t} + \frac{2}{3} \tan t \Big|_0^{\pi/3}$$

$$= \frac{\sqrt{3}}{3} \cdot 4 + \frac{2}{3}\sqrt{3} = \sqrt{3}\left(\frac{4}{3} + \frac{2}{3}\right) = \underline{2\sqrt{3}}$$

$$\therefore I = \frac{1}{4}\left(\pi/3 - \left(\pi/6 + \frac{\sqrt{3}}{8}\right) + \frac{1}{16}\sqrt{3} - \frac{1}{16} \cdot 2\sqrt{3}\right)$$

$$= \frac{1}{4}\left(\pi/6 + \sqrt{3}\left(-\frac{1}{8} + \frac{1}{16} - \frac{1}{8}\right)\right)$$

$$= \underline{\frac{\pi}{24} - \frac{3\sqrt{3}}{64}}$$