Oppgaver:

6.5 1abd, 2, 3, 7, 12

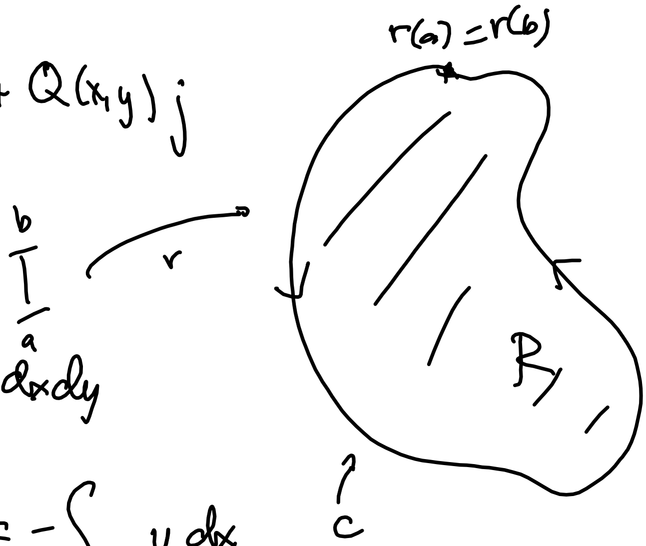
Midtveiseksamen 2018

Greens teorem

$$\mathbb{F}(x,y) = P(x,y)\, i + Q(x,y)\, j$$

Green:

$$\int_C F \cdot dr = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$



$$\text{Areal}(R) = \int_C x\,dy = -\int_C y\,dx \quad \Leftarrow \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

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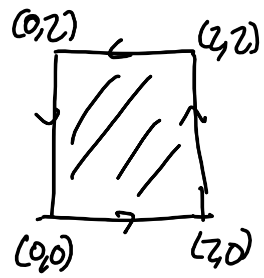
1 a)

$$I = \int_C (x^2+y)\,dx + x^2 y\,dy$$



$$Q = x^2 y \qquad \therefore \quad \frac{\partial Q}{\partial x} = 2xy$$

$$P = x^2 + y \qquad \frac{\partial P}{\partial y} = 1$$

$$\leadsto I = \iint_R 2xy - 1 \; dx\,dy = \int_0^2 \int_0^2 2xy - 1 \; dx\,dy$$

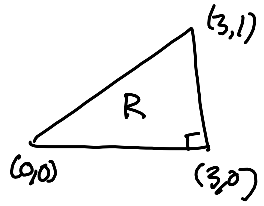
$$= \int_0^2 x^2 y - x \Big|_0^2 \, dy$$

$$= \int_0^2 4y - 2 \; dy$$

$$= 2y^2 - 2y \Big|_0^2 = 8 - 4 = \underline{4}$$

b

$$\oint x^2 y^3\,dx + x^3 y^2\,dy$$

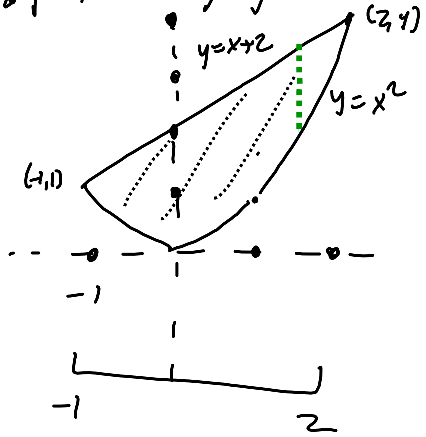


$$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$$= \iint_R 3x^2y^2 - 3x^2y^2\,dx\,dy = \underline{0}$$

d)

$$\int_C (x^2 y + x e^x)\,dx + (xy^3 + e^{\sin y})\,dy$$



$$= \iint_R y^3 - x^2\,dx\,dy$$

$$= \int_{-1}^{2}\int_{x^2}^{x+2} y^3 - x^2\,dy\,dx$$

$$= \int_{-1}^{2} \frac{y^4}{4} - x^2 y \Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} \frac{(x+2)^4}{4} - x^2(x+2) - \frac{x^8}{4} + x^4\,dx$$

$$= \int_{-1}^{2} -\frac{x^8}{4} + \frac{5}{4}x^4 + x^3 + 4x^2 + 8x + 4\,dx$$

$$= -\frac{x^9}{36} + \frac{1}{4}x^5 + \frac{x^4}{4} + \frac{4x^3}{3} + 4x^2 + 4x \Big|_{-1}^{2}$$

$$= -\frac{512}{36} + \frac{32}{4} + \frac{16}{4} + \frac{32}{3} + 16 + 8$$

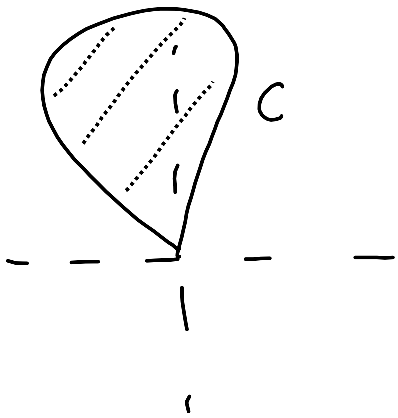
$$- \left(-\frac{1}{36} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \frac{4}{3} + \cancel{4} - \cancel{4}\right)$$

$$= \underline{\frac{135}{4}}$$

$\underline{2}$ $\quad r(t) = (t \sin t)\, i + (2\pi t - t^2)\, j$

$t \in [0, 2\pi]$

$$A = \int_C x\, dy$$



$$dy = (2\pi - 2t)\, dt$$

$$A = \int_C (t \sin t)(2\pi - 2t)\, dt$$

$$= 2\pi \int_0^{2\pi} t \sin t\, dt - 2 \int_0^{2\pi} t^2 \sin t\, dt$$

$\Big[$ $\int t \sin t\, dt = -\cos t \cdot t + \sin t$ (delvis integrasjon)

$\int t^2 \sin t\, dt = -t^2 \cos t + 2t \sin t + \cos t$

$\Big\lfloor$

$$A = 2\pi \left(\sin t - \cos t \cdot t \,\Big|_0^{2\pi} \right) - 2 \left(-t^2 \cos t + 2t \underline{\sin t} + \underline{\cos t} \right) \Big]_0^{2\pi}$$

$$= -2\pi \cdot 2\pi - 2\left(-(2\pi)^2 \right) = -(2\pi)^2 + 2(2\pi)^2 = \underline{4\pi^2}$$

5 Regn ut $\iint_R x \, dx \, dy$ R = området avgrenset av kurven C

C: $r(t) = (t - t^2)\,i + (t - t^3)\,j$
$t \in [0,1]$

Green:

$$I = \iint_R x \, dx = -\int_C xy \, dx \qquad dx = (1-2t)\,dt$$

$P = xy \quad \frac{\partial P}{\partial y} = -x$
$Q = 0$

$$\therefore \; I = -\int_0^1 (t-t^2)(t-t^3)(1-2t)\,dt$$

$$= -\int_0^1 (1-2t)(t^2 - t^3 - t^4 + t^5)\,dt$$

$$= -\int_0^1 t^2 - t^3 - t^4 + t^5 - 2t^3 + 2t^4 + 2t^5 - 2t^6 \, dt$$

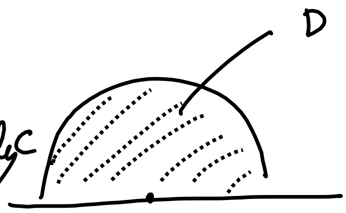
$$= -\int_0^1 t^2 - 3t^3 + t^4 + 3t^5 - 2t^6 \, dt$$

$$= -\left[\tfrac{1}{3} - \tfrac{3}{4} + \tfrac{1}{5} + \tfrac{3}{6} - \tfrac{2}{7} \right]$$

$$= \underline{\frac{1}{420}}$$

7. $D =$ området i $\mathbb{R}^2$ av punkter (x,y) s.a.

$$x^2 + y^2 \leq 1 \quad \text{og} \quad y \geq 0$$



$$I = \oint_C \left(xy + \ln(x^2+1)\right)dx + \left(4x + e^{y^2} + 3\arctan y\right)dy$$

$$\overset{\text{Green}}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dxdy = \iint_D (4 - x)\, dx\, dy$$

$$= \int_0^{\pi}\int_0^{1} (4 - r\cos\theta)\, r \, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{1} 4r - r^2\cos\theta \, dr\, d\theta$$

$$= \int_0^{1}\int_0^{\pi} 4r - r^2\cos\theta \, d\theta\, dr$$

$$= \int_0^{1} 4r\theta - r^2\sin\theta \Big|_0^{\pi} dr$$

$$= \int_0^{1} 4r\pi \, dr = 4\pi\int_0^1 r\, dr = 4\pi \cdot \frac{r^2}{2}\Big|_0^1 = \frac{4\pi}{2} = \underline{\underline{2\pi}}$$

12 E: $9x^2+4y^2-18x+16y=11$

a) Finn senter og halvakser til E

b) Vis at $r(t)=(1+2\cos t)i+(-2+3\sin t)j$ er en parametrisering av E.

Regn ut $\oint_E F\cdot dr$ der

$F = y^2 i + x j$

c) Regn ut $\iint_R 1-2y \, dx\,dy$ R = området avgrenset av E.

$$9x^2+4y^2-18x+16y=11$$
$$9(x-1)^2-9+4(y+2)^2-16=11$$
$$9(x-1)^2+4(y+2)^2=36$$
$$\frac{(x-1)^2}{36/9}+\frac{(y+2)^2}{36/4}=1$$
$$\frac{(x-1)^2}{2^2}+\frac{(y+2)^2}{3^2}=1$$

halvakser: $a=2$, $b=3$

Senter: $(1,-2)$

b) Vis at $r(t)$ er en parametrisering:

Vet at $(\cos t, \sin t)$ er en parametrisering av sirkelen $u^2+v^2=1$,

$\rightsquigarrow$ anvend lineærtransformasjonen $\mathbb{R}^2 \to \mathbb{R}^2$, $(u,v)\xmapsto{T}(\underbrace{1+2u}_{x}, \underbrace{-2+3v}_{y})$

$x=1+2u$, $y=-2+3v$; $u=\frac{x-1}{2}$, $v=\frac{y+2}{3}$

$u^2+v^2=1 \Leftrightarrow \left(\frac{x-1}{2}\right)^2+\left(\frac{y+2}{3}\right)^2=1$ ∴ C

$\rightsquigarrow T(\cos t,\sin t)=(1+2\cos t, -2+3\sin t)$

$\Rightarrow r(t)$ er en parametrisering av C.

$I=\oint_C F\cdot dr = \int_0^{2\pi}$

$F=y^2 i + x j$ $\quad r'(t)=(-2\sin t, 3\cos t)$

$$F\cdot dr = \begin{pmatrix}(-2+3\sin t)^2\\ 1+2\cos t\end{pmatrix}\cdot\begin{pmatrix}-2\sin t\\ 3\cos t\end{pmatrix}$$

$$= 3\cos t + 6\cos^2 t - 8\sin t + 24\sin^2 t - 18\sin^3 t$$

$\int_0^{2\pi}=0$ $\quad \int_0^{2\pi}=0$ $\quad \int_0^{2\pi}=0$

$I=\int_0^{2\pi} F\cdot dr$

$$\rightsquigarrow \frac{1}{6}I = \int_0^{2\pi}\cos^2 t + 4\sin^2 t\,dt$$
$$= \int_0^{2\pi} 1+3\sin^2 t\,dt \qquad \sin^2 t+\cos^2 t=1$$
$$= 2\pi + 3\int_0^{2\pi}\sin^2 t\,dt \qquad \sin^2 t = \frac{1}{2}-\frac{1}{2}\cos 2t$$
$$= 2\pi + 3\int_0^{2\pi}\frac{1}{2}-\frac{1}{2}\cos 2t\,dt$$
$$= 2\pi + 3\left[\frac{t}{2}-\cos t\sin t\right]_0^{2\pi} \qquad \sin 2t = 2\cos t\sin t$$
$$= 2\pi + 3\pi = 5\pi$$

$\rightsquigarrow \frac{1}{6}I=5\pi \rightsquigarrow I=30\pi$

c) $\iint_R (1-2y)\,dx\,dy \underset{\text{Green}}{=} \int_C F\cdot dr = 30\pi$

$P=y^2$, $Q=x$ $\rightsquigarrow \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=1-2y$

Midtveis 2019

$$S = \begin{cases} 2x + 2y = 1 \\ x + 3y = 3 \\ 6x + 10y = a \end{cases}$$ entydig løsning?

$$S \text{ entydig løsning} \iff A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 3 \\ 6 & 10 & a \end{pmatrix} \sim \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 3 \\ 6 & 10 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 3 \\ 2 & 2 & 1 \\ 6 & 10 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 3 \\ 0 & -4 & -5 \\ 0 & -8 & a-18 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & a-8 \end{pmatrix}$$

$\therefore$ S entydig løsning $\iff \underline{a = 8}$.

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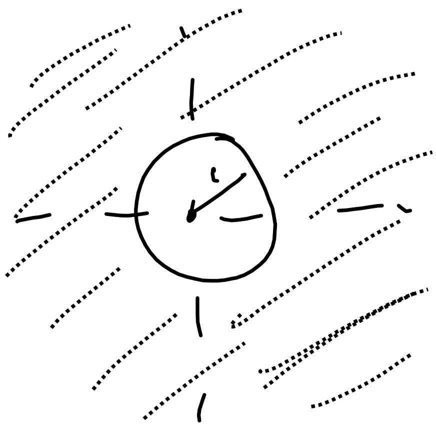
$$\iint_R \frac{dx\,dy}{(x^2+y^2)^3} \qquad R = \{x^2+y^2 \geq 1\}$$



$$= \lim_{R\to\infty} \int_0^{2\pi}\int_1^{R} \frac{r}{(r^2)^3}\,dr\,d\theta$$

$$= \lim_{R\to\infty} 2\pi \int_1^{R} r^{-5}\,dr$$

$$= \lim_{R\to\infty} 2\pi \frac{1}{-5+1} r^{-6}\Big|_1^R$$

$$= \lim_{R\to\infty} 2\pi \cdot \left(\frac{1}{-4} \underbrace{R^{-6}}_{R\to\infty\ 0} + \frac{1}{4} \right) = \frac{1}{4}\cdot 2\pi = \underline{\pi/2}$$