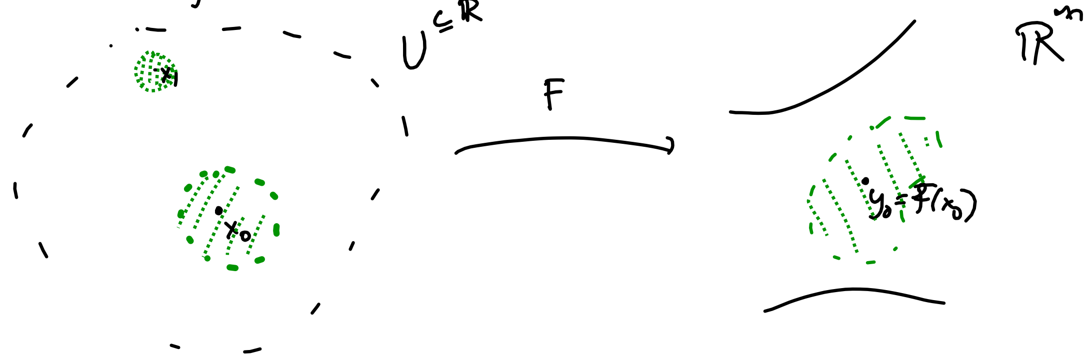
5.7

Oppgaver: 1, 2, 3, 4, 5
7, 9, 10, 11

Invers funksjons teorem



- F er C^1 ($\frac{\partial F}{\partial x_i}$ kontinuerlige for alle i)
- $F'(x)$ invertibar for alle $x \in U$.

$\Rightarrow$ Da finnes en omegn $U_0 \ni x_0$ s.a $F|_{U_0}$ er bijektiv.

$\therefore$ $\exists$ invers $G: V \longrightarrow U_0$ (V omegn om $y_0 = F(x_0)$)

s.a $F(G(y)) = y$ $G(F(x)) = x$ for alle $x \in U_0$, $y \in V$

G er deriverbar i V og

$$G'(y_0) = F'(x_0)^{-1}$$

ex $f(x) = x^2$ på U

$U = \mathbb{R}$: $f(x)$ er ikke injektiv på noen omegn om 0 (x og $-x$ må være med i U)

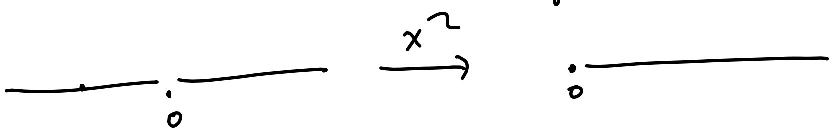
Merk: $f'(0) = 2x|_{x=0} = 0$

$U = \mathbb{R}_{>0}$: $f'(x) \neq 0$ for $x \in U$

IFT $\rightsquigarrow$ f har en lokal invers rundt hvert punkt x_0 ($g(y) = \sqrt{y}$)

$U = \mathbb{R} - 0$: $f'(x) = 0$ for alle $x \in U$

men f er ikke injektiv på U



$G(F(x)) = x$ (antar vi har vist at G fines og er C^1)

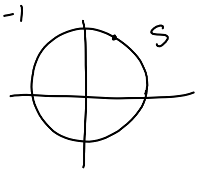
$\Rightarrow G'(F(x)) \cdot F'(x) = I_m$

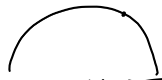
$\Rightarrow G'(F(x)) = F'(x)^{-1}$ ✓

Implisitt funksjonsteorem

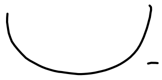
$f(x,y) = x^2 + y^2 - 1$

$S = \{ f(x,y) = 0 \}$

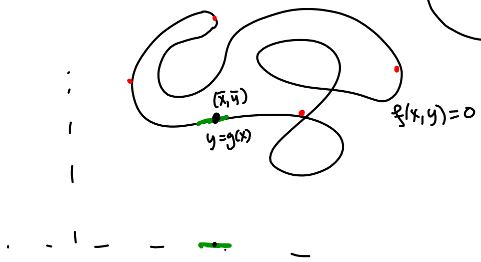




grafen til $\sqrt{1-x^2}$



grafen til $-\sqrt{1-x^2}$



Gitt $f: \mathbb{R}^{m+1} = \underset{x}{\mathbb{R}^m} \times \underset{y}{\mathbb{R}} \longrightarrow \mathbb{R}$

- f er C^1
- $(\bar{x}, \bar{y})$ $\quad f(\bar{x}, \bar{y}) = 0$
- $\frac{\partial f}{\partial y}(\bar{x}, \bar{y}) \neq 0$

$\Rightarrow$ Det fines en omegn $U_0 \ni (\bar{x}, \bar{y})$ s.a $\{ f(x,y) = 0 \}$ lokalt ser ut som grafen til en funksjon nær $(\bar{x}, \bar{y})$

$$\{ f(x,y) = 0 \} \cap U_0 = \{ f(x, g(x)) = 0 \}$$
$(y = g(x))$

$$\frac{\partial g}{\partial x_i}(\bar{x}) = - \frac{\frac{\partial f}{\partial x_i}(\bar{x}, \bar{y})}{\frac{\partial f}{\partial y}(\bar{x}, \bar{y})}$$

$f(x, g(x)) = 0$ for alle $x \in U_0$:

$$\Rightarrow \frac{\partial f}{\partial x_i}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x)) \frac{\partial g}{\partial x_i}(x) = 0$$

$\Rightarrow$ løs for $\frac{\partial g}{\partial x}$ og få formelen ovenfor

Vektorvaluert versjon:

$U \subseteq \mathbb{R}^{m+k}$ $\quad (\bar{x}, \bar{y}) \in U$

$F: U \longrightarrow \mathbb{R}^k$ $\quad C^1$

- $F(\bar{x}, \bar{y}) = 0$
- $\frac{\partial F}{\partial y}(\bar{x}, \bar{y})$ inverterbar ($k \times k$ matrise)

$\Rightarrow \exists$ omegn $U_0 \ni \bar{x}$ og $G: U_0 \longrightarrow \mathbb{R}^k$

s.a $F(x, G(x)) = 0$ for alle $x \in U_0$.

$$G'(x) = -\left(\frac{\partial F}{\partial y}(\bar{x}, \bar{y}) \right)^{-1} \left(\frac{\partial F}{\partial x}(\bar{x}, \bar{y}) \right).$$

1. $F(x,y) = (x^2 + y + 1 \; , \; x - y - 2)$. Vis at det fins en $G: \mathbb{R}^2 \to \mathbb{R}^2$ definert nær $(1,-2)$ s.a $G(1,-2) = (0,0)$.

Finn $G'(1,-2)$.

Vis at F har en omvendt funksjon H definert i en omegn om $(1,-2)$ s.a $H(1,-2) = (-1,-1)$.

Finn $H'(1,-2)$.

$(\bar{x},\bar{y}) = (0,0)$: $\quad F'(0,0) = \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}\Bigg|_{\substack{x=0\\y=0}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

Denne er inverterbar : $\det F'(0,0) = \underline{-1}$.

IFT $\Rightarrow$ $\exists U_0 \ni (0,0)$ + G invers til F definert i en omegn av $(1,-2) = F(0,0)$.

$$G'(1,-2) = F'(0,0)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{\det}\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}$$

La nå $(\bar{x},\bar{y}) = (-1,-1)$: $\quad F(-1,-1) = \begin{pmatrix} 1-1+1 \\ -1+1-2 \end{pmatrix} = \underline{(1,-2)}$

$$F'(-1,-1) = \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}\Bigg|_{\substack{x=-1\\y=-1}} = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$$

Inverterbar: $\det F'(-1,-1) = 2 - 1 = 1$

IFT $\Rightarrow$ finnes en lokal invers $G: V \to U_0$ for F i en omegn $V \ni (1,-2)$.

$$G'(1,-2) = F(-1,-1)^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \underline{\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}}$$

2 $F(x,y) = \left(e^{x+y^2-1}, x-y\right) : \mathbb{R}^2 \to \mathbb{R}^2$

$(x_0, y_0) = (0,1)$. Skal vise at $\exists$ invers $G: V \to V_0$

der V omegn om $(1,-1)$

V_0 omegn om $(0,1)$

og finne $G'(1,-1)$.

F er C^1 ✓

Regner ut $F'(0,1)$:

$$F'(0,1) = \begin{pmatrix} e^{x+y^2-1} & 2y\, e^{x+y^2-1} \\ 1 & -1 \end{pmatrix}_{\substack{x=0 \\ y=1}}$$

$$= \begin{pmatrix} e^0 & 2e^0 \\ 1 & -1 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}}$$

Denne er inverterbar: $\det F'(0,1) = -1-2 = \underline{-3}$

$\to$ IFT gir en invers funksjon $G: V \to U_0$ som over.

$$G'(1,-1) = F'(0,1)^{-1} = \frac{1}{-3}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix} = \underline{\frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}}$$

For $\bar{x} = (-3,-2)$:

$$F'(-3,-2) = \begin{pmatrix} e^{-3+4-1} & -4e^0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 1 & -1 \end{pmatrix}$$

Inverterbar ✓ $\det = -1+4 = 3$

$$G'(1,-1) = \begin{pmatrix} 1 & -4 \\ 1 & -1 \end{pmatrix}^{-1} = \underline{\frac{1}{3}\begin{pmatrix} -1 & 4 \\ -1 & 1 \end{pmatrix}}$$

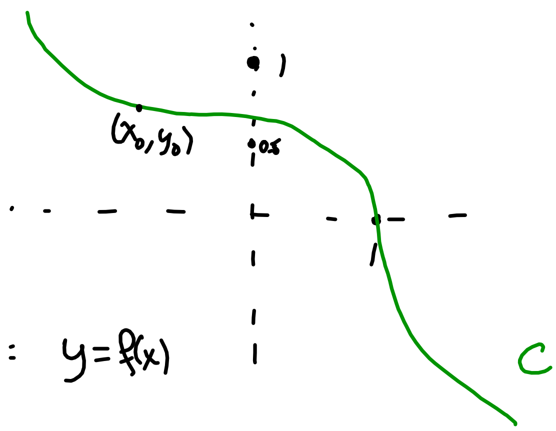
3 C: $x^3 + y^3 + y = 1$

Vis at gjennom hvert punkt (x_0,y_0) på C går det en funksjon $y = f(x)$ som tilfredsstiller ligningen. Finn $f'(x_0)$.

$$F(x,y) = x^3 + y^3 + y - 1$$

$$F'(x,y) = (3x^2,\ 3y^2+1)$$

$$\therefore\ \frac{\partial F}{\partial y} = 3y^2 + 1 \neq 0$$



Implisitt funksjonsteoremet gjør resten: $y = f(x)$ i en omegn om (x_0, y_0).

$$f'(x_0) = -\frac{\frac{\partial f}{\partial x}(x_0,y_0)}{\frac{\partial f}{\partial y}(x_0,y_0)} = \frac{-3x_0^2}{3y_0^2+1}$$

<u>4</u> $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ $(x_0, y_0, z_0) = (-1, 2, 0)$

$f(x,y,z) = xy^2 e^z + z$ $f(x_0, y_0, z_0) = -4$

Vis at $\exists$ omegn U_0 om (x_0, y_0) + $g(x,y)$ s.a

$f(x,y,g(x,y)) = -4$

$$\frac{\partial f}{\partial z}(-1,2,0) = xy^2 e^z + 1 \Big|_{(-1,2,0)} = -4e^0 + 1 = -3 \neq 0$$

IFT
$\Rightarrow$ $\exists\, g = g(x,y)$ definert om $(-1,2)$ s.a $f(x,y,g(x,y)) = \underline{-4}$

$$\frac{\partial g}{\partial x}(-1,2) = \frac{-\frac{\partial f}{\partial x}(x_0, y_0, z_0)}{\frac{\partial f}{\partial z}(x_0, y_0, z_0)} \Bigg|_{(-1,2,0)} = \frac{-y^2 e^z}{-3} \Bigg|_{(-1,2,0)}$$

$$= \underline{4/3}$$

$$\frac{\partial g}{\partial y} = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}} \Bigg|_{(-1,2,0)} = \frac{-2xye^z}{-3} = \underline{-\frac{4}{3}}$$

5

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

a) Finn A^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{array}\right) \underset{\mathrm{III}-\mathrm{I}}{\overset{\mathrm{II}-2\mathrm{I}}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -3 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 0 & -1/3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 1 & 0 & -5/3 & 1 & -1/3 \\ 0 & 0 & 1 & 1/3 & 0 & -1/3 \end{array}\right)$$

$$\therefore \quad A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 1 \\ -5 & 3 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

b) $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$F(x,y,z) = \begin{pmatrix} x+z \\ x^2 + \frac{1}{2}y^2 + z \\ x + z^2 \end{pmatrix}$$

Vis at F har en omvendt funksjon G definert, en omegn om $(0, \frac{1}{2}, 2)$ s.a $G(0,\frac{1}{2},2) = (1,1,-1)$. Finn $G'(0,\frac{1}{2},2)$.

I $(x_0,y_0,z_0) = (1,1,-1)$:

$$F'(1,1,-1) = \left.\begin{pmatrix} 1 & 0 & 1 \\ 2x & y & 1 \\ 1 & 0 & 2z \end{pmatrix}\right|_{\substack{x=1 \\ y=1 \\ z=-1}} = \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}}_{= A}$$

Vet at A er invertibel $\overset{\text{IFT}}{\Rightarrow}$ Det finnes en lokal invers som over.

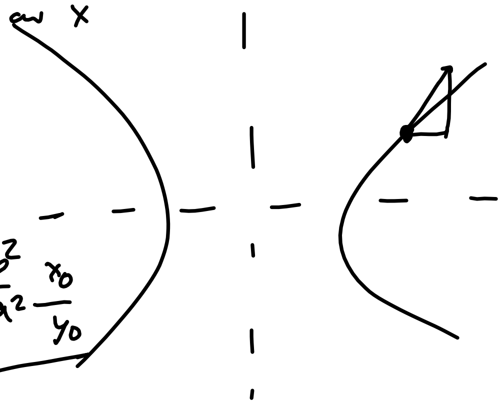
$$G'(0, \tfrac{1}{2}, 2) = F'(1,1,-1)^{-1} = A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 1 \\ -5 & 3 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

7 $H : F(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Finn stigningstallet til H i $(x_0, y_0) \in H$ $(y_0 \neq 0)$.

Kan ved IFT se på y som en funksjon av x

$\left(\frac{\partial F}{\partial y} = -\frac{2y}{b^2} \neq 0 \right)$ $\underline{y = y(x)}$



Ved IFT:

$$\text{stigningstallet} = -\left.\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right|_{(x_0,y_0)} = \frac{-\frac{2x}{a^2}}{-\frac{2y}{b^2}} = \frac{b^2}{a^2}\frac{x_0}{y_0}$$

9 $\quad \phi(x, y(x)) = C \qquad C$ konstant.

Vis at $\quad y'(x) = \dfrac{-\dfrac{\partial \phi}{\partial x}(x, y(x))}{\dfrac{\partial \phi}{\partial y}(x, y(x))}$

gitt at $\dfrac{\partial \phi}{\partial x}(x, y(x)) \neq 0$ (og at alle partiellderiverte eksisterer).

$\phi(x, y(x)) = C$ for alle x

$$\to \quad 0 = \frac{\partial}{\partial x}\phi(x, y(x)) = \frac{\partial}{\partial x}\phi(x, y(x)) + \frac{\partial}{\partial y}\phi(x, y(x)) \cdot y'(x)$$

$$\leadsto \quad y'(x) = -\frac{\frac{\partial}{\partial x}\phi(x, y(x))}{\frac{\partial}{\partial y}\phi(x, y(x))}$$

10 En funksjon $z = z(x,y)$ tilfredsstiller likningen

$$x + y^2 + z^3 = 3xyz \qquad (*)$$

Finn $\frac{\partial z}{\partial x}$ og $\frac{\partial z}{\partial y}$.

$$F(x,y,z) = x + y^2 + z^3 - 3xyz \qquad (*) \Leftrightarrow F(x,y,z) = 0$$

$$\frac{\partial F}{\partial x} = 1 + 3z^2 \frac{\partial z}{\partial x} - 3yz - 3xy \frac{\partial z}{\partial x} = 0$$

$$(1 - 3yz) + \frac{\partial z}{\partial x}(3z^2 - 3xy) = 0 \Rightarrow \frac{dz}{dx} = \frac{3yz - 1}{3z^2 - 3xy}$$

$$0 = \frac{\partial F}{\partial y} = 2y + 3z^2 \frac{\partial z}{\partial y} - 3xz - 3xyz \frac{\partial z}{\partial y}$$

$$\rightsquigarrow \quad \frac{\partial z}{\partial y} = \frac{3xz - 2y}{3z^2 - 3xyz}.$$

<u>11</u> $z = z(x,y)$ tilfredsstiller likningen

$$2x^2 + 3y^2 + z^2 = e^{-z}$$

Finn $\frac{\partial z}{\partial x}$ og $\frac{\partial z}{\partial y}$

$$F(x,y,z) = 2x^2 + 3y^2 + z^2 - e^{-z}$$

$$0 = \frac{\partial F}{\partial x} = 4x + 2z\frac{\partial z}{\partial x} + e^{-z}\frac{\partial z}{\partial x} = 0$$

$$\rightsquigarrow \frac{\partial z}{\partial x} = \frac{-4x}{2z + e^{-z}}$$

$$0 = \frac{\partial F}{\partial y} = 6y + 2z\frac{\partial z}{\partial y} + e^{-z}\frac{\partial z}{\partial y} = 0$$

$\rightsquigarrow$ løser for $\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial y} = \frac{-6y}{2z + e^{-z}}$$