Oppgaver:

12.1 4abce, 5

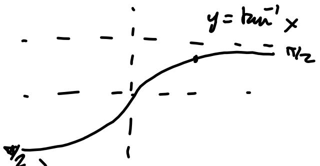
12.2 1abe, 2, 3abdf, 5, 6, 7, 9

12.1
- $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ $|r|<1$.
- "Divergenstesten": $\sum_{n=0}^{\infty} a_n$ konvegerer $\Rightarrow \lim_{n\to\infty} a_n = 0$.

4 a) $\sum_{n=0}^{\infty} \arctan n$ divergerer:

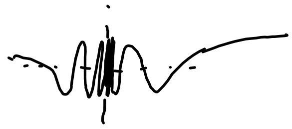
$\arctan x \to \pi/2$ når $x \to \infty$.

$\left(\tan x = \frac{\sin x}{\cos x} \quad \underset{=0}{\cos \pi/2}\right)$



b) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

$y = \cos(\frac{1}{x})$:



$\cos(\frac{1}{n}) \to 1$ når $n \to \infty$

$\to$ divergerer.

c) $\sum_{n=1}^{\infty} \underbrace{\left(1 - \sin\left(\frac{1}{n}\right)\right)^n}_{a_n}$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \sin\left(\frac{1}{n}\right)\right)^n$ $\sin x = x - \frac{x^3}{3!} + \dots$

$= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \underline{e^{-1}}$

e) $\sum_{n=1}^{\infty} n\left(\sqrt[n]{2} - 1\right)$

$f(x) = 2^x$ $\frac{f(x)-f(0)}{x-0} = f'(c)$ $c \in (0,x)$ ← Middelverdisetningen

$\frac{f(\frac{1}{n}) - f(0)}{1/n} = \frac{2^{1/n} - 1}{1/n} = n(2^{1/n} - 1)$ $f'(x) = 2^x \log 2$

$\leadsto \lim_{n\to\infty} a_n = \lim_{n\to\infty} f'(c_n) = \lim_{n\to\infty} 2^{1/n} \log 2 = \underline{\log 2}$

$c_n \in (0, \frac{1}{n})$

5 $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

a) Delbrøksoppspalting på $\frac{1}{k(k+1)}$:

$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} \leadsto 1 = A(k+1) + Bk$

$k=0$: $1 = A\cdot 1 + 0\cdot B \Rightarrow A = 1$

$k=-1$: $1 = A\cdot 0 + (-1)B \Rightarrow B = -1$

$\leadsto \frac{1}{k(k+1)} = \underline{\frac{1}{k}} - \underline{\frac{1}{k+1}}$

b) $\underbrace{\sum_{k=1}^{n} \frac{1}{k(k+1)}}_{S_n} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \dots - \frac{1}{n+1}$

$= 1 - \frac{1}{n+1}$

c) S_n konvegerer mot 1 $\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$

12.2 $\qquad S = \sum_{n=1}^{\infty} a_n \qquad \underline{a_n > 0}$

Konvergenstester:

- S konvergerer $\Leftrightarrow s_n = \sum_{k=1}^{n} a_k$ begrenset følge
- Integraltesten: $f: [1, \infty) \to \mathbb{R}$ kontinuerlig, positiv, avtagende:
  (IT) $\sum_{n=1}^{\infty} f(n)$ konvergerer $\Leftrightarrow \int_1^{\infty} f(x)\,dx$ konvergerer
  $\hookrightarrow \sum \frac{1}{n^p}$ konvergerer hviss $p > 1$
- Sammenligningstesten: $\sum a_n$, $\sum b_n$ positive rekker
  (ST) (i) $b_n \leq c \cdot a_n$ og $\sum a_n$ konvergerer $\Rightarrow \sum b_n$ konvergerer
  (ii) $b_n \geq c \cdot a_n$ og $\sum a_n$ divergerer $\Rightarrow \sum b_n$ divergerer
  $c > 0$
- Grensesammenligningstesten: (i) $\sum a_n$ konvergerer og $\lim_{n\to\infty} \frac{b_n}{a_n} < \infty$
  (GST) $\Rightarrow \sum b_n$ konvergerer
  (ii) $\sum a_n$ divergerer og $\lim \frac{b_n}{a_n} < \infty \Rightarrow \sum b_n$ divergerer.
- Forholdstesten: $\sum a_n$
  (FT) $a = \lim \frac{a_{n+1}}{a_n}$ eksisterer (kan ha $a = \infty$)
  $a < 1 \Rightarrow \sum a_n$ konvergerer
  $a = 1 \Rightarrow$ ingen konklusjon
  $a > 1 \Rightarrow \sum a_n$ divergerer
- Rottesten: $a = \lim \sqrt[n]{a_n}$ tilsvarende som

1a Integraltesten: $f(x) = \frac{1}{1+x}$ kontinuerlig, avtagende og positiv (*)

$\sum_{n=0}^{\infty} \frac{1}{n+1}$: $\lim_{y\to\infty} \int_1^y \frac{dx}{1+x} = \lim_{y\to\infty} \log(1+y) = \infty$

$\rightsquigarrow$ rekken divergerer.

b) $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ $f(x) = \frac{1}{x^2+1}$ tilfredsstiller (*)

$\lim_{y\to\infty} \int_1^y \frac{dx}{x^2+1} = \lim_{y\to\infty} \arctan y = \pi/2$ $\rightsquigarrow$ rekka konvergerer

c) $\sum_{n=0}^{\infty} \left(\pi/2 - \arctan n\right)$ → leddene går mot 0.

$f(x) = \pi/2 - \arctan x$ tilfredsstiller betingelsen (*) :

$I = \lim_{y\to\infty} \int_1^y f(x)\,dx = \lim_{y\to\infty} \int_1^y \pi/2 - \arctan x \; dx$

$\int \arctan x \, dx \overset{D.I}{=} x \arctan x - \int \frac{x}{x^2+1}\,dx$ $\quad t = x^2+1$, $dt = 2x\,dx$

$u = \arctan x$ $\quad v = x$

$du = \frac{1}{1+x^2}dx$ $\quad dv = dx$ $\quad = x \arctan x - \frac{1}{2}\log(x^2+1)$.

$\Rightarrow I = \underbrace{\pi/2\, x - x\arctan x}_{x(\pi/2 - \arctan x)} + \frac{1}{2}\log(x^2+1) \Big|_1^{\infty}$ $\to \infty$ "0/0"

$\lim_{x\to\infty} x(\pi/2 - \arctan x) = \lim_{x\to\infty} \frac{\pi/2 - \arctan x}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} = 1$

$\rightsquigarrow I = \lim_{y\to\infty} \frac{1}{2}\log(x^2+1)\Big|_1^y = \infty$

IT: Rekken divergerer.

2. Bruk IT til å vise at
$$\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$$
konvergerer hvis $p>1$ og divergerer for $p \leq 1$.

$f(x) = \dfrac{1}{x(\log x)^p}$ kontinuerlig positiv avtagende $\quad p \neq 1:$

$$\int_1^{\infty} \frac{1}{x(\log x)^p}\,dx = \lim_{y\to\infty} \int_0^y u^{-p}\,du = \lim_{y\to\infty} \frac{1}{-p+1} u^{-p+1} \Big|_0^y$$

$u = \log x$

$du = \frac{1}{x}\,dx$

$$= \lim_{y\to\infty} \frac{1}{-p+1} y^{1-p} = \begin{cases} 0 & p>1 \\ \infty & p<1 \end{cases}$$

$p=1$: $\displaystyle\int_1^{\infty} \frac{1}{x\log x}\,dx = \int_0^{\infty} \frac{du}{u} = \log u \Big|_0^{\infty} = \infty.$

$\leadsto$ IT viser at $\sum \frac{1}{n(\log n)^p}$ konvergerer kun for $p>1$, og divergerer ellers.

3 oblig:

$a_n = 1/n \quad b_n = \frac{7n^2+3}{4n^3-2}$

a) $\sum_{n=1}^{\infty} \frac{7n^2+3}{4n^3-2} \Rightarrow \frac{b_n}{a_n} = \frac{\frac{7n^2+3}{4n^3-2}}{\frac{1}{n}} = \frac{7n^3+3n}{4n^3-2} \xrightarrow[n\to\infty]{} \frac{7}{4}$

GST $\Rightarrow$ rekka divergerer ($\sum \frac{1}{n}$ diverger)

$\Rightarrow \int_1^{\infty} \frac{dx}{x} = \infty$

$\Rightarrow$ Integraltesten

$\Rightarrow \sum \frac{1}{n}$ diverger

b) $\sum_{n=1}^{\infty} \frac{2n-7}{4n^3+8} \quad a_n = \frac{1}{n^2} \leadsto \frac{b_n}{a_n} = \frac{\frac{2n-7}{4n^3+8}}{\frac{1}{n^2}} = \frac{2n^3-7}{4n^3+8} \xrightarrow[n\to\infty]{} 1/2$

$\leadsto$ GST $\Rightarrow$ rekka konvergerer (siden $\sum \frac{1}{n^2}$ konvergerer).

d) $\sum_{n=1}^{\infty} \frac{\cos(\frac{1}{n})}{n+\sqrt{n}} =: b_n$, $\cos(\frac{1}{n}) \xrightarrow[n\to\infty]{} 1$

dominerer $\Rightarrow$ sammenligner med $1/n$

$a_n = 1/n$:

$\frac{b_n}{a_n} = \frac{\frac{\cos(\frac{1}{n})}{n+\sqrt{n}}}{\frac{1}{n}} = \frac{\cos(\frac{1}{n})}{1+\sqrt{n}/n} \to \frac{1}{1}$, $\cos(\frac{1}{n}) \to 1$, $\sqrt{n}/n \to 0$

GST $\Rightarrow$ rekka divergerer (da $\sum \frac{1}{n}$ divergerer)

f) $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$, $\sin(\frac{1}{n}) \to 0$

$\sim \frac{1}{n}$ når $n \to \infty$

$\left(\frac{\sin x}{x} \to 1\right)$

Sammenligner med $a_n = \frac{1}{n}$:

$\frac{b_n}{a_n} = \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \to 1$

GST $\Rightarrow$ rekka divergerer.

5 Forholdstesten / Rottesten:

a) $\sum_{n=1}^{\infty} n \cdot \frac{1}{3^n}$ $\quad a = \lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{(n+1)}{3^{n+1}}}{\frac{n}{3^n}} = \lim \frac{n+1}{3n} = \frac{1}{3}$

$\leadsto$ rekka konverger ved FT.

c) $\sum_{n=1}^{\infty} \underbrace{\left(1-\frac{1}{n}\right)^{n^2}}_{a_n}$ $\quad a = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n$

$= \underline{e^{-1}} < 1$

konvergerer ved rottesten.

e) $\sum_{n=1}^{\infty} \frac{2^n}{\underline{n^n}}$ $\quad a = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{\cancel{2^{n+1}}}{(n+1)^{n+1}}}{\frac{\cancel{2^n}}{n^n}} = \lim_{n\to\infty} \frac{2n^n}{(n+1)^{n+1}} = 0$

$\rightarrow$ rekka konvergerer ved FT.

g) $\sum_{n=1}^{\infty} \frac{n!\, 4^n}{n^n}$ $\quad a = \lim \frac{\frac{(n+1)\cancel{!}\cancel{4^{n+1}}}{(n+1)^{n+1}}}{\frac{\cancel{n!}\cancel{4^n}}{n^n}} = \lim \frac{(n+1)\, 4\, n^n}{(n+1)^{n+1}}$

$(n+1)! = n! \cdot (n+1)$

$= \lim_{n\to\infty} \frac{4n^n}{(n+1)^n} \rightarrow 4$

$a > 1 \Rightarrow$ rekka divergerer.

6. a) $\sum \frac{n}{n^2+1}$

Prøver FT: $\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)}{(n+1)^2+1}}{\frac{n}{n^2+1}} = \frac{(n+1)\cdot(n^2+1)}{((n+1)^2+1)\,n}$

$= \frac{n^3+\dots}{n^3+\dots} \xrightarrow[n\to\infty]{} 1$

$\rightsquigarrow$ FT gir ingen informasjon.

GST med $\frac{1}{n}$: $\frac{a_n}{\frac{1}{n}} = \frac{n/(n^2+1)}{\frac{1}{n}} = \frac{n^2}{n^2+1} \longrightarrow 1$ når $n\to\infty$

$\sum \frac{1}{n}$ divergerer $\Rightarrow \sum \frac{n}{n^2+1}$ divergerer ved GST.

b) $\sum_{n=1}^{\infty} \underbrace{\sinh\left(\frac{1}{n}\right)}_{a_n}$ $\qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$

$\frac{a_{n+1}}{a_n} = \frac{e^{\frac{1}{n+1}} - e^{-\frac{1}{n+1}}}{e^{1/n} - e^{-1/n}} = \frac{(e^{2/n+1} - 1)\,e^{-\frac{1}{n+1}}}{\underbrace{(e^{2/n} - 1)}\,\underbrace{e^{-1/n}}}$

Middelverdisetningen: $f(x) = e^x$

$\frac{e^{2/(n+1)} - 1}{e^{2/n} - 1} = \frac{f'(c_1)}{f'(c_2)}$ $\quad c_1, c_2 \in \left(0, \frac{2}{n}\right)$

$\Rightarrow \lim \frac{e^{2/n+1}-1}{e^{2/n}-1} = \lim \frac{e^{1/n}}{e^{1/n}} = 1.$

$\rightarrow a = \lim \frac{a_{n+1}}{a_n} = 1 \quad \rightsquigarrow$ FT ingen informasjon.

Prøver GST med $\frac{1}{n}$:

$e^x = 1 + x + \frac{x^2}{2!} + \dots$

$e^x - e^{-x} = \left(1 + x + \frac{x^2}{2!} + \dots\right) - \left(1 - x + x^2 - \frac{x^3}{3!} + \dots\right) = 2x + \frac{2x^3}{3!} + \dots$

$\frac{\sinh\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{e^{1/n} - e^{-1/n}}{2\frac{1}{n}}$

$= \frac{2\cdot\left(\frac{1}{n}\right) + \frac{2}{3!}\left(\frac{1}{n}\right)^3 + \dots}{2\frac{1}{n}}$

$= 1 + \frac{1}{3!}\left(\frac{1}{n}\right)^2 + \dots \longrightarrow 1$ når $n\to\infty$

$\sum \sinh\left(\frac{1}{n}\right)$ divergerer ved GST.

c) $\sum \left(1 - \cosh\left(\frac{1}{n}\right)\right)$ $\qquad \cosh x = \frac{e^x + e^{-x}}{2}$

$1 - \cosh(x) = 1 - \frac{e^x + e^{-x}}{2} = 1 - \frac{1}{2}\left[1 + \cancel{x} + \frac{x^2}{2!} + \dots + 1 - \cancel{x} + \frac{x^2}{2!} \dots\right]$

$= -\frac{x^2}{2} - \frac{x^4}{24} - \dots$

$\therefore 1 - \cosh\left(\frac{1}{n}\right) \approx -\frac{1}{2}\left(\frac{1}{n}\right)^2$ når $n >> 0$:

Sammenligner med $\frac{1}{n^2}$:

$\frac{1 - \cosh\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \frac{-\frac{1}{2}\left(\frac{1}{n}\right)^2 - \cancel{\frac{1}{24}\left(\frac{1}{n}\right)^4} + \dots}{\frac{1}{n^2}}$

$\longrightarrow -\frac{1}{2}$ når $n\to\infty$.

$\therefore$ Rekka konvergerer ved GST.

7.

a) $\sum \frac{7}{n^2+2n+3}$ konvergerer ved sammenligning med $\frac{1}{n^2}$:

$$\frac{\frac{7}{n^2+2n+3}}{\frac{1}{n^2}} = \frac{7n^2}{n^2+2n+3} \longrightarrow 7 \quad \text{eksisterer}$$

$\Rightarrow$ konvergens.

b) $\sum \frac{\sqrt{n}}{1+n}$ GST med $\frac{1}{\sqrt{n}}$:

$$\frac{\frac{\sqrt{n}}{1+n}}{\frac{1}{\sqrt{n}}} = \frac{n}{1+n} \longrightarrow 1 \quad \text{når } n \to \infty$$

$\Rightarrow$ rekka divergerer ($\sum \frac{1}{\sqrt{n}}$ divergerer)

c) $\sum \frac{\ln n}{n}$ $\quad \frac{\ln n}{n} > \frac{1}{n} \Rightarrow$ rekka divergerer

d) $\sum \left(1+\frac{1}{n}\right)^n$ her går ikke leddene mot 0:

$\left(1+\frac{1}{n}\right)^n > 1 \Rightarrow$ divergerer.

e) $\sum n e^{-n^2}$

FT:

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)e^{-(n+1)^2}}{n e^{-n^2}} = \lim \frac{n+1}{n} e^{-(n+1)^2+n^2}$$

$$= \lim e^{-2n-1} = 0$$

9): $\sum \frac{1+2+\dots+n}{2^n}$ konvergerer:

$$\sum_{n} \frac{\frac{n(n+1)}{2}}{2^n} = \sum \frac{n(n+1)}{2^{n+1}}$$

Forholdstesten:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)(n+2)}{2^{n+2}}}{\frac{n(n+1)}{2^{n+1}}} = \overbrace{\frac{(n+1)(n+2)}{n(n+1)}}^{\to 1} \cdot \frac{1}{2} \longrightarrow \frac{1}{2}$$

$\rightsquigarrow a < 1 \Rightarrow$ rekka konvergerer.