Oppgaver:

4.3 1, 2, 3, 4, 5, 6, 7

4.4 1, 2, 3, 4, 5

4.5 1, 2ab, 3a, 4a, 6, 7

4.6 1, 2, 3ab, 4, 6, 7abcd, 8ab, 9, 10a

Recap

M matrix: $n \times n$

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

trappeform $\rightsquigarrow$

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Redusert trappeform $\rightsquigarrow$

$$\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} x + y &= 1 \\ 2x + y - z &= 4 \end{aligned} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & -1 & 4 \end{pmatrix}$$

trappeform $\rightsquigarrow$ $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 3 & -2 \end{pmatrix}$

redusert trappeform $\rightsquigarrow$ $\begin{pmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & 3 & -2 \end{pmatrix}$

fri variabel

$$x - 3z = 3$$
$$y + 3z = -2$$

lettere å løse

$$x = 3 + 3z$$
$$y = -2 - 3z \qquad z \in \mathbb{R}$$

2a

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \underset{\mathrm{II}-3\mathrm{I}}{\sim} \begin{pmatrix} 1 & 2 \\ 0 & 5-3\cdot 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \overset{\mathrm{I}-2\mathrm{II}}{\sim} \underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

trappeform ↗

b)

$$B = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 1 & 1 & 0 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

$$\underset{\mathrm{II}-2\mathrm{I}}{\sim} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & 3 & -2 \end{pmatrix} \underset{(-1)\,\mathrm{II}}{\sim} \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & -3 & 2 \end{pmatrix}$$

$$\sim \underline{\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & 2 \end{pmatrix}}.$$

c)

$$C = \begin{pmatrix} 1 & 1 & 2 & 2 \\ -1 & 2 & -1 & 3 \\ 2 & 2 & 1 & 1 \end{pmatrix}.$$

$$\underset{\mathrm{III}-2\mathrm{I}}{\overset{\mathrm{I}+\mathrm{II}}{\sim}} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -3 & -3 \end{pmatrix} \underset{(\div 3)\,\mathrm{III}}{\sim} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$C=\begin{pmatrix} 1 & 1 & 1 & 2 \\ -1 & 2 & -1 & 3 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

$$\underset{\substack{\mathrm{II}+\mathrm{I} \\ \mathrm{III}+(-2)\mathrm{I}}}{\sim}\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 3 & 0 & 5 \\ 0 & 0 & -1 & -3 \end{pmatrix}$$

$$\sim\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\overset{\mathrm{I}-\mathrm{II}}{\sim}\begin{pmatrix} 1 & 0 & 1 & 2-5/3 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1/3 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\sim\begin{pmatrix} 1 & 0 & 0 & -8/3 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

<u>4</u>

$$x + 2y + z = b_1$$
$$2x + 4y + 3z = b_2$$
$$-x + 3y + 2z = b_3$$

Q: har dette en løsning x, y, z?

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$$

hvis $(A, b) \sim (I, x)$

så har systemet en løsning.

$$A \underset{\substack{II - 2 \cdot I \\ III + I}}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 5 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3/5 \\ 0 & 0 & 1 \end{pmatrix}$$

Denne har pivotelementer i hver kolonne $\Rightarrow$ entydig løsning for alle b_1, b_2, b_3.

4.4.1 $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Finn $\vec{x}$ s.a. $A\vec{x} = b$.

Utvidet matrise: $\begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$ (A, b)

$$\sim \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \end{pmatrix} \underset{II-2\cdot I}{\sim} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -4 \end{pmatrix} \overset{I-II}{\sim} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \end{pmatrix} = (I, x)$$

$\rightsquigarrow$

$$\underline{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}}.$$

4.4.1 b

$$A=\begin{pmatrix}1&1&1\\-1&2&-1\\2&-2&1\end{pmatrix}\qquad b=\begin{pmatrix}2\\2\\0\end{pmatrix}$$

Finn $x=\begin{pmatrix}x\\y\\z\end{pmatrix}$ s.a $Ax=b$.

$$\left(\begin{array}{cccc}1&1&1&2\\-1&2&-1&2\\2&-2&1&0\end{array}\right)\underset{\mathrm{III}+(-2)\mathrm{I}}{\overset{\mathrm{I}+\mathrm{II}}{\sim}}\left(\begin{array}{cccc}1&1&1&2\\0&3&0&4\\0&-4&-1&-4\end{array}\right)$$

$$\sim\left(\begin{array}{cccc}1&1&1&2\\0&1&0&4/3\\0&4&1&4\end{array}\right)$$

$$\underset{\mathrm{III}-4\,\mathrm{II}}{\sim}\left(\begin{array}{cccc}1&0&1&2-4/3\\0&1&0&4/3\\0&0&1&4-16/3\end{array}\right)\sim\left(\begin{array}{cccc}1&0&1&2/3\\0&1&0&4/3\\0&0&1&-4/3\end{array}\right)$$

$$\sim\left(\begin{array}{cccc}1&0&0&2\\0&1&0&4/3\\0&0&1&-4/3\end{array}\right)\sim\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\4/3\\-4/3\end{pmatrix}$$

4.4.7 $C = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & a^2-a & 3 \\ -1 & 1 & -3 & a \end{pmatrix}$ $a \in \mathbb{R}$

a) redusere C til trappeform

b) $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & a^2-a \\ -1 & 1 & -3 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 3 \\ a \end{pmatrix}$

Q: for hvilke $a \in \mathbb{R}$ har $Ax = b$ entydig løsning / ingen løsning / uendelig mange løsninger?

a) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & a^2-a & 3 \\ -1 & 1 & -3 & a \end{pmatrix} \underset{\substack{II-2I \\ III+I}}{\sim} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & a^2-a-2 & 1 \\ 0 & 1 & -2 & a+1 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & a+1 \\ 0 & 0 & a^2-a & -a \end{pmatrix}$ trappeform

b) Ser at om $a^2 - a \neq 0 \implies$ pivotelement i hver kolonne $\implies$ entydig løsning.

$a^2 - a = a(a-1)$:

Om $a=0$: siste rad blir $(0\ 0\ 0\ 0)$ $\leadsto$ uendelig mange løsninger.

Om $a=1$ $\leadsto$ siste rad er $(0\ 0\ 0\ -1)$

$0x + 0y + 0z = 1$ $\leadsto$ ingen løsning.

4.5

① $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$

Finner A^{-1} ved å radredusere (A, I)

$$\rightsquigarrow (I, A^{-1})$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{pmatrix} \underset{II - 3I}{\sim} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -7 & -3 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/7 & -1/7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1/7 & 2/7 \\ 0 & 1 & 3/7 & -1/7 \end{pmatrix}$$

$$\rightsquigarrow \underline{A^{-1} = \frac{1}{7} \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}}$$

b) $A = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$

$$\left(\begin{array}{cccc} 1 & -2 & 1 & 0 \\ -3 & 6 & 0 & 1 \end{array}\right) \underset{II+3I}{\sim} \left(\begin{array}{cccc} 1 & -2 & 1 & 0 \\ \underbrace{0 \quad 0} & & 3 & 1 \end{array}\right)$$

→ ikke mulig å for I ..

→ matrisen A er ikke invertibar.

$\underline{6}$ $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}$

a) Finn B^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{\mathrm{III}+2\mathrm{II}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2/3 & 1/3 \end{array}\right)$$

$$\underset{\substack{\mathrm{I}-2\mathrm{II} \\ \mathrm{II}-\mathrm{III}}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 2/3 & 1/3 \end{array}\right)$$

$$\overset{\mathrm{I}+2\mathrm{III}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2+4/3 & 2/3 \\ 0 & 1 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 2/3 & 1/3 \end{array}\right)$$

$$\Rightarrow B^{-1} = \frac{1}{3}\begin{pmatrix} 3 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

b) Bruk dette til å løse $Bx = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$

$$\therefore x = B^{-1}\begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 3\cdot 5 - 2\cdot\cancel{3} + 2/3 \\ 0\cdot 5 + 3\cdot 1 - 3\cdot 1 \\ 0\cdot 5 + 2\cdot 3 + 1\cdot 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$$

c) For hvilke $a, b \in \mathbb{R}$ har $\overline{\text{ligningssystemet}}$

$$x + 2y = 5$$
$$y + z = 3$$
$$-2y + (a+1)z = b^2 - 10$$

Én; ingen; eller ∞ mange løsninger?

$$\left(B + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ b^2 - 10 \end{pmatrix}$$

Gange begge sider med B^{-1}:

$$B^{-1}B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 & 0 & 2a \\ 0 & 0 & -a \\ 0 & 0 & a \end{pmatrix}$$

$$B^{-1}\left(B + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2a/3 \\ 0 & 1 & -a/3 \\ 0 & 0 & a+1/3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{?}{=} \begin{pmatrix} 5 \\ 3 \\ b^2 \end{pmatrix}$$

$\underline{a + 1/3 \neq 0}$ $(\Leftrightarrow a \neq -3) \Rightarrow$ entydig løsning.

$\underline{a = -3}$:

$$y + z = 3$$

$$\cdot(-2) \curvearrowright \quad -2y - 2z = b^2 - 10$$

(Sett inn: ligningssystemet)

$$y + z = -\frac{1}{2}(b^2 - 10) = 3$$

$$b^2 - 10 = -6 \Leftrightarrow b^2 = 4 \Leftrightarrow b = \pm 2$$

4.5.7

Anta A er en inverterbar $n \times n$ matrise og at b er en radvektor med n komponenter. Vis at $x = bA^{-1}$ er den entydige løsningen i ligningen $xA = b$.

Sjekker at $x = bA^{-1}$ er en løsning:

$$xA = (bA^{-1})A = b(A^{-1}A) = bI = b$$

$\rightsquigarrow$ OK.

Entydig: Dersom y er en annen løsning: $yA = b$

$\rightsquigarrow$ $(x-y)A = b - b = 0$ $\rightsquigarrow$ $v = x - y$ er s.a.

$vA = 0$ $\rightsquigarrow$ $\exists$ lineærkombinasjon mellom søylene i A

$\rightsquigarrow$ A ikke er inverterbar

$\rightsquigarrow$ motsigelse om ikke $v = 0$. $\square$

4.6.1 $b = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

$A = (a_1\ a_2) = \begin{pmatrix} 1 & -4 \\ -2 & 2 \end{pmatrix}$

Skriv b som en lineær kombinasjon av a_1 og a_2.

$\Leftrightarrow \quad \lambda_1 a_1 + \lambda_2 a_2 = b \quad \Leftrightarrow \quad A\lambda = b$ ← likningssystem

utvidet matrise:

$$\begin{pmatrix} 1 & -4 & -3 \\ -2 & 2 & 3 \end{pmatrix} \underset{II+2I}{\sim} \begin{pmatrix} 1 & -4 & -3 \\ 0 & -6 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & -3 \\ 0 & 1 & 1/2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \end{pmatrix}$$

λ

$-a_1 + \frac{1}{2} a_2 = b$: $\quad -\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} = b$

$\triangleright$

9 Avgjør om følgende vektorer utspenner en basis:

a) $\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}$

Basis for $\mathbb{R}^m$: lineært uavhengig mengde vektorer $a_1 \dots a_n$
s.a $Sp(a_1, \dots, a_n) = \mathbb{R}^m$

$\Rightarrow \underline{m = n}$

Her er $m = 2$, $n = 3 \Rightarrow$ ikke en basis.

b) $\begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}4\\1\\2\end{pmatrix}$ for $\mathbb{R}^3$

To vektorer kan ikke utspenne hele $\mathbb{R}^3 \Rightarrow$ ikke en basis,

c) $\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}$ $\mathbb{R}^2$

Sjekke om disse er lineært uavhengige (+ utspenner $\mathbb{R}^2$)

Disse vektorene er ikke proporsjonale $\Rightarrow$ lineært uavhengige.

d) $\begin{pmatrix}-1\\3\\-2\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\3\\2\end{pmatrix}$ i $\mathbb{R}^3$

lineært uavhengige:

$$\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix} = \begin{pmatrix}-1 & 3 & 2\\ 2 & 0 & 1\\ -1 & 3 & -2\end{pmatrix} \sim \begin{pmatrix}-1 & 3 & 2\\ 2 & 0 & 1\\ 0 & 0 & 4\end{pmatrix}$$

$\rightsquigarrow$ pivotsøyler i hver kolonne

$\rightsquigarrow$ lineært uavhengige. $\square$