Forelesning, 27. januar 2021

onsdag 27. januar 2021 12:13

1. Repetisjon
2. Kjeglesnitt
   - parabel
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3. Dandelins iskrembevis

Eks. $\bar{G} = (e^y + 2x,\ xe^y + 2yz,\ y^2) = (G_1, G_2, G_3)$

sjekker kryssderivate:

$$\frac{\partial G_1}{\partial y} = e^y \qquad \frac{\partial G_1}{\partial z} = 0 \qquad \frac{\partial G_2}{\partial z} = 2y$$

$$\parallel \qquad\qquad \parallel \qquad\qquad \parallel$$

$$\frac{\partial G_2}{\partial x} = e^y \qquad \frac{\partial G_3}{\partial x} = 0 \qquad \frac{\partial G_3}{\partial y} = 2y$$

Gode sjanser for at $\bar{G} = \nabla f$. Finn f.

$$\begin{tikzcd}
 & f \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[d, "\frac{\partial}{\partial y}"] \arrow[dr, "\frac{\partial}{\partial z}"] & \\
e^y + 2x & xe^y + 2yz & y^2
\end{tikzcd}$$

$$e^y + 2x = \frac{\partial f}{\partial x}$$

$$\int e^y + 2x \, dx = e^y \cdot x + x^2 + h(y,z) \overset{?}{=} f(x,y,z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^y x + x^2 + h(y,z)\right) = xe^y + \frac{\partial h}{\partial y} \overset{?}{=} xe^y + 2yz$$

Krever

Går foreløpig bra dersom $\frac{\partial h}{\partial y} = 2yz$

$$h(y,z) = \int 2yz \, dy = y^2 z + g(z)$$

Revidert kandidat for f:

$$f(x,y,z) = e^{y}\cdot x + x^2 + y^2 z + g(z)$$

$$\frac{\partial f}{\partial z} = y^2 + g'(z) \underset{\substack{\uparrow \\ \text{krever}}}{=} y^2 \quad \text{dvs } g'(z) = 0$$

eller $g(z) = C$

Det gir $\underline{\underline{f(x,y,z) = e^{y}x + x^2 + y^2 z + C}}$

Eks

$$G_1 = yz \qquad \frac{\partial G_1}{\partial y} = z \qquad \frac{\partial G_1}{\partial z} = y \qquad \frac{\partial G_2}{\partial z} = x$$
$$G_2 = xz \qquad\qquad \shortparallel \qquad\qquad \shortparallel \qquad\qquad \shortparallel$$
$$G_3 = xy \qquad \frac{\partial G_2}{\partial x} = z \qquad \frac{\partial G_3}{\partial x} = y \qquad \frac{\partial G_3}{\partial y} = x$$

Kryssderivasjon ok. Finn f slik at
$\bar{G} = \nabla f$

$$\begin{tikzcd} & f \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[d, "\frac{\partial}{\partial y}"] \arrow[dr, "\frac{\partial}{\partial z}"] & \\ yz & xz & xy \end{tikzcd}$$

$$f = \int yz\,dx = xyz + h(y,z)$$

$$\frac{\partial f}{\partial y} = xz + \frac{\partial h}{\partial y} \underset{\substack{\uparrow \\ \text{krever}}}{=} xz \quad \text{dvs } \frac{\partial h}{\partial y} = 0 \quad h = h(z)$$

Reviderer kandidaten: $f(x,y,z) = xyz + h(z)$

$$\frac{\partial f}{\partial z} = xy + h'(z) \underset{\substack{\uparrow \\ \text{krever}}}{=} xy \quad \text{dvs } h'(z) = 0$$

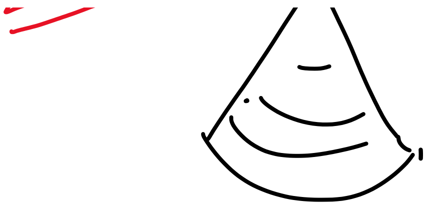
eller $h(z) = C$

$$\Longrightarrow \underline{\underline{f(x,y,z) = xyz + C}}$$

2. Kjeglesnitt

Snittene er kjeglesnitt

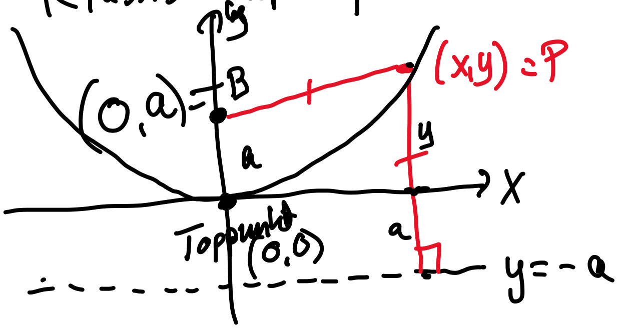
Avhengig av vinkelen på planet (og kjeglen) vil snittet være en
- ellipse (sirkel)



– parabel
– hyperbel

Parabel ($y=x^2$)

Klassisk definisjon



B: Brennpunkt

Andre muligheter:

- $y^2=4bx$
- $(x-\alpha)^2=4a(y-\beta)$

Mengden av punkter P slik at $|BP|$ = avstanden fra P til linja $y=-a$

$$|BP|=\|(x,y)-(0,a)\|$$
$$=\sqrt{x^2+(y-a)^2}$$

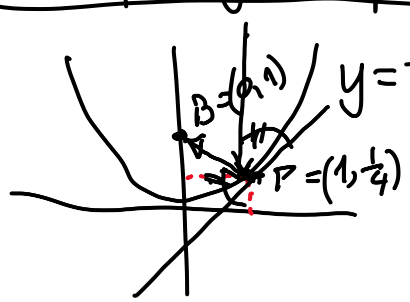
Fra P til linja: $y+a$

Vil ha $\sqrt{x^2+(y-a)^2}=y+a$

$$x^2+\cancel{y^2}-2ay+\cancel{a^2}=\cancel{y^2}+2ay+\cancel{a^2}$$

$x^2=4ay$ dvs $\underline{y=\frac{1}{4a}x^2}$

Refleksjonsegenskapen til en parabel



$y=\frac{1}{4}x^2$
$\Downarrow$
$y'=\frac{1}{2}x$

Tangent i $P=(1,\frac{1}{4})$

$$y-\frac{1}{4}=\frac{1}{2}(x-1)$$

eller $y=\frac{1}{2}x-\frac{1}{4}$

Enhetstangent: $\dfrac{(1,\frac{1}{2})}{\sqrt{1+(\frac{1}{2})^2}}=\dfrac{(2,1)}{\sqrt{5}}$

BP: $B=(0,1)$

$$y-1=\frac{\frac{1}{4}-1}{1-0}(x-0) \quad \text{dvs } y=-\frac{3}{4}x+1$$

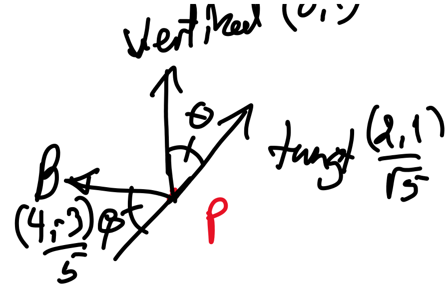
Enhetstangent $\dfrac{(1,-\frac{3}{4})}{\sqrt{1^2+(-\frac{3}{4})^2}}=\dfrac{(4,-3)}{5}$

Vertikal linje, tangent $(0,1)$

Innfallsvinkel = utgangsvinkel

Vinkel mellom vektorer $\bar{u}, \bar{v}$

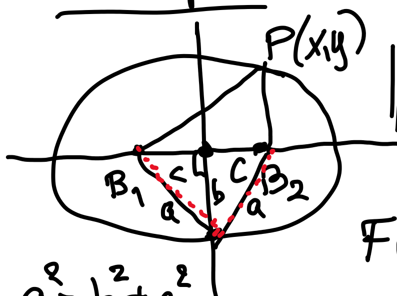
$$\frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\| \cdot \|\bar{v}\|} = \cos\theta$$



$$\frac{(2,1)}{\sqrt{5}} \cdot (0,1) = \frac{1}{\sqrt{5}} = \cos\theta$$

$$\frac{(4,-3)}{5} \frac{(2,1)}{\sqrt{5}} = \frac{5}{5\sqrt{5}} = \frac{1}{\sqrt{5}} = \cos\varphi$$

Ellipse



$a^2 = b^2 + c^2$

$\|B_1P\| + \|B_2P\| = \text{konstant}$ (uavhengig av P)
$= 2a$

Finne en likning for ellipsen

$B_1 = (-c, 0)$ $\quad \|B_1P\| = \sqrt{(x-(-c))^2 + (y-0)^2} = \sqrt{(x+c)^2 + y^2}$

$B_2 = (c, 0)$ $\quad \|B_2P\| = \sqrt{(x-c)^2 + (y-0)^2} = \sqrt{(x-c)^2 + y^2}$

$P = (x, y)$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2} \qquad \text{Kvadreren}$$

$$\cancel{x^2} + 2xc + \cancel{c^2} + \cancel{y^2} = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + \cancel{x^2} - 2xc + \cancel{c^2} + \cancel{y^2}$$

$$\cancel{4}a\sqrt{(x-c)^2 + y^2} = \cancel{4}a^2 - \cancel{4}xc \qquad \text{Kvadrer}$$

$$\underbrace{a^2(x^2 - 2xc + c^2 + y^2)} = a^4 - \cancel{2a^2xc} + x^2c^2$$

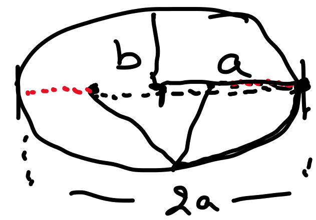
$$= x^2a^2 - \cancel{2xca^2} + a^2c^2 + a^2y^2$$

$$x^2a^2 - x^2c^2 + a^2y^2 = a^4 - a^2c^2 \qquad a^2 - c^2 = b^2$$

$$b^2x^2 + a^2y^2 = a^2b^2 \qquad \frac{1}{a^2b^2}$$

b: lille halvakse
a: store –"–

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$



Eks

$$9x^2 + 4y^2 - 36x + 24y + 36 = 0$$

$$9(x^2 - 4x) + 4(y^2 + 6y) = -36$$

$$9(\underbrace{x^2 - 4x + 4}_{(x-2)^2}) + 4(\underbrace{y^2 + 6y + 9}_{(y+3)^2}) = -36 + 36 + 36 = 36$$

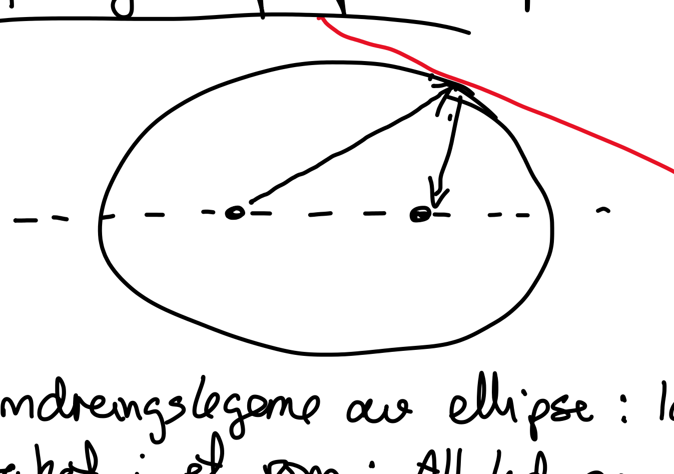
$$9(x-2)^2 + 4(y+3)^2 = 36 \quad \Big| \frac{1}{36}$$

Ellipse med halvakser 2 og 3

$$\frac{(x-2)^2}{2^2} + \frac{(y+3)^2}{3^2} = 1$$

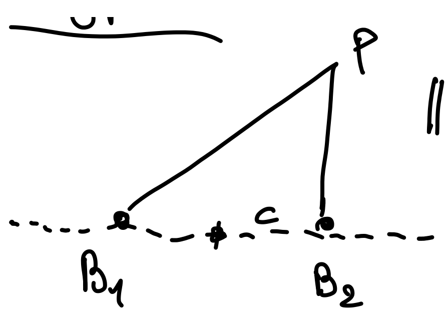
Sentrum = (2, −3)

Refleksjonsegenskap for ellipse:



Omdreiningslegeme av ellipse: lar det være taket i et rom: All lyd som sendes ut fra det ene brennpunktet havne i det andre.

Hyperbel



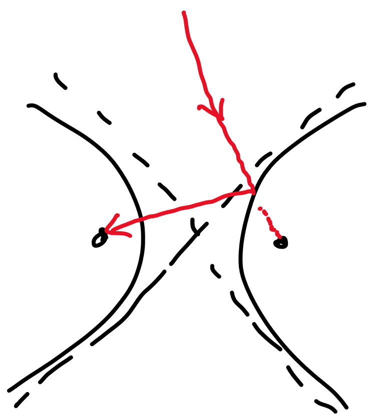
$$\|B_1P\| - \|B_2P\| = 2a$$

Alle P som oppfyler dette gir en hyperbel

setter: $b = \sqrt{c^2 - a^2}$

$$\leadsto \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

standardform for hyperbel



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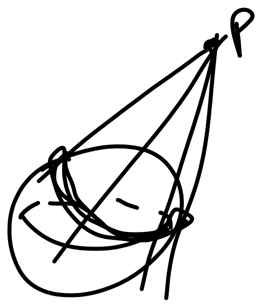
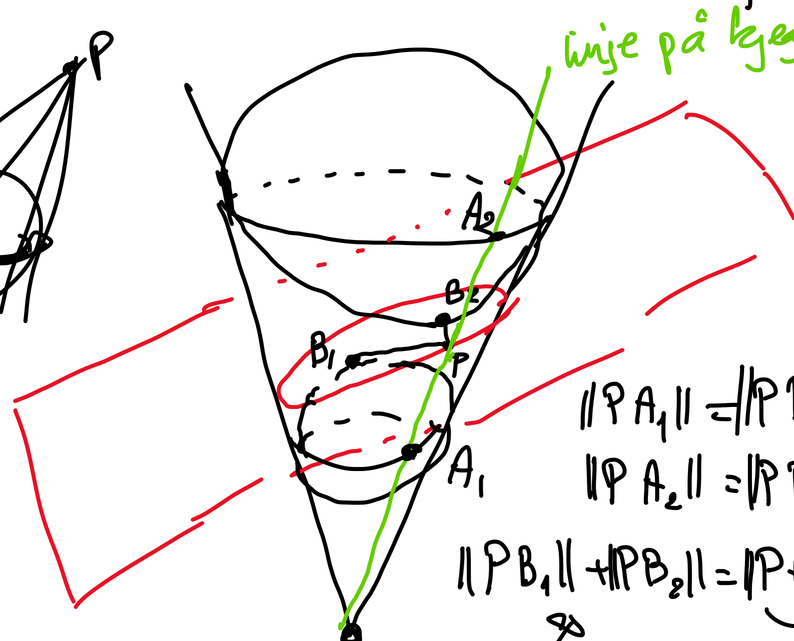
Dandelins iskrem bevis

Knytter sammen en ellipse (summen av avstandene til brennpunktene er konstant) med ellipsen som et kjeglesnitt (snitt mellom et plan og en kjegle)





$$\|PA_1\| = \|PB_1\|$$

$$\|PA_2\| = \|PB_2\|$$

$$\|PB_1\| + \|PB_2\| = \underbrace{\|PA_1\| + \|PA_2\|}_{= \|A_1A_2\|}$$

uavhengig av P

uavhengig av .

dvs konstant. uavhengig av P