Forelesning, 1. februar 2021

mandag 1. februar 2021 14:13

3.3 2. $f(x,y) = xy$ $\mathcal{C}$: $\bar{r}(t) = 3t\,\bar{i} + 4t\,\bar{j}$

$= (3t, 4t)$

$0 \leq t \leq 2.$

$\bar{r}'(t) = (3, 4)$

$\|\bar{r}'(t)\| = \sqrt{3^2 + 4^2} = 5$

$$\int_{\mathcal{C}} f\,ds = \int_0^2 f(3t,4t) \cdot 5 \cdot dt$$

$$= \int_0^2 12t^2 \cdot 5 \; dt$$

$$= 60 \int_0^2 t^2\,dt = 60\left[\tfrac{1}{3}t^3\right]_0^2 = \frac{60 \cdot 8}{3} - 0 = \underline{\underline{160}}$$

3. $f(x,y,z) = z \cdot \cos(xy)$ $\mathcal{C}$: $\bar{r}(t) = (3t, 4t, 5t)$

$0 \leq t \leq \sqrt{\pi}$

$\bar{r}'(t) = (3, 4, 5)$

$\|\bar{r}'(t)\| = \sqrt{3^2 + 4^2 + 5^2}$

$= \sqrt{9 + 16 + 25}$

$= \sqrt{50} = \underline{5 \cdot \sqrt{2}}$

$$\int_{\mathcal{C}} f\,ds = \int_0^{\sqrt{\pi}} 5t \cos(12t^2) 5 \cdot \sqrt{2}\,dt$$

$$= 25\sqrt{2} \int_0^{\sqrt{\pi}} t \cdot \cos(12t^2)\,dt$$

$u = 12t^2$

$du = 24t\,dt$

$t = 0 \Rightarrow u = 0$

$t = \sqrt{\pi} \Rightarrow u = 12\pi.$

$t = \sqrt{\pi}$

$$= 25\sqrt{2} \int \cos(\underbrace{12t^2}_{u}) \underbrace{24t\,dt}_{du} \frac{1}{24}$$

$t=0$

$$= 25\sqrt{2} \int_{u=0}^{u=12\pi} \cos(u) \frac{1}{24} du$$

$$= \frac{25}{24}\sqrt{2} \Big[\sin(u)\Big]_0^{12\pi} = \frac{25}{24}\sqrt{2}\left(\sin(12\pi) - \sin(0)\right) = \underline{\underline{0}}.$$

11. $C: \bar{r}(t) = \left(\frac{t^2}{2}, \frac{2\sqrt{2}}{9} t^{\frac{3}{2}}, \frac{t}{9}\right)$ $\quad 1 \leq t \leq 7$

$z = \frac{t}{9}$, $dz = \frac{1}{9} dt$

$$\int_C \left(\frac{1}{15} + \frac{1}{2}\frac{dz}{ds}\right) ds = \text{Totalt bensinforbruk.}$$

$\frac{1}{15} + \frac{1}{2}\frac{dz}{ds}$: bensinforbruk pr. lengde

ds: bitteliten lengde

$$\bar{r}'(t) = \left(t, \frac{2\sqrt{2}}{9}\cdot\frac{3}{2} t^{\frac{3}{2}-1}, \frac{1}{9}\right)$$

$$= \left(t, \frac{\sqrt{2}}{3} t^{\frac{1}{2}}, \frac{1}{9}\right)$$

$$\|\bar{r}'(t)\| = \sqrt{t^2 + \frac{2}{9}t + \frac{1}{81}}$$

$$= \sqrt{\left(t + \frac{1}{9}\right)^2}$$

$$= t + \frac{1}{9}$$

$$= \int_1^7 \frac{1}{15}\left(t + \frac{1}{9}\right) dt + \int_1^7 \frac{1}{2} \frac{\frac{1}{9}dt}{ds} ds$$

$$= \int_1^7 \frac{1}{15}t + \frac{1}{9\cdot 15} + \frac{1}{18} \, dt$$

$$= \left[\frac{1}{15\cdot 2}t^2 + \frac{1}{135}t + \frac{1}{18}t\right]_1^7$$

$$= \frac{49}{30} + \frac{7}{135} + \frac{7}{18} - \frac{1}{30} - \frac{1}{135} - \frac{1}{18} = \frac{89}{45}$$

12. $r = f(\theta) \qquad a \le \theta \le b$

a) $\vec{r}(\theta) = (x(\theta), y(\theta)) = (r\cdot\cos(\theta), r\sin\theta)$

$$= \left(f(\theta)\cos\theta,\ f(\theta)\sin(\theta)\right) \qquad a \le \theta \le b$$

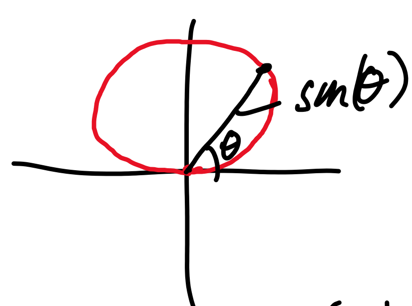
b) $\|\vec{r}\,'(\theta)\| = \|(f'(\theta)\cdot\cos\theta - f(\theta)\sin\theta,\ f'(\theta)\cdot\sin\theta + f(\theta)\cos\theta)\|$

$$= \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin(\theta))^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}$$

$$= \left(f'(\theta)^2\cdot\cos^2\theta - 2f'(\theta)f(\theta)\cos\theta\sin\theta + f(\theta)^2\sin^2(\theta) + f'(\theta)^2\sin^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta + f(\theta)^2\cos^2\theta\right)^{1/2}$$

$$= \sqrt{f'(\theta)^2 + f(\theta)^2}$$

c) $r = f(\theta) = \sin(\theta) \qquad 0 \le \theta \le \pi$



$f(\theta) = \sin(\theta)$

Ers.

$$B = \int_C ds = \int_0^{\pi} \|\vec{r}\,'(\theta)\|\cdot d\theta$$

$$= \int_0 \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta \qquad f'(\theta) = \cos(\theta)$$

$$= \int_0^{\pi} 1 \cdot d\theta = [\theta]_0^{\pi} = \underline{\underline{\pi}}$$

d) $g(x,y) = xy$

$x = f(\theta) . \cos\theta = \sin\theta\cos\theta$
$y = f(\theta) . \sin\theta = \sin^2\theta$

$$\int_C g\, ds = \int_0^{\pi} \sin(\theta) \cdot \cos(\theta) \cdot \sin^2(\theta) \cdot 1 \cdot d\theta$$

$$= \int_0^{\pi} \sin^3\theta \cdot \cos\theta\, d\theta$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$$= \int_0^0 u^3\, du = \underline{\underline{0}}$$

$\theta = 0 \Rightarrow u = 0$
$\theta = \pi \Rightarrow u = 0$

3.4)

2. $\bar{F}(x,y) = (x^2, xy) = x^2\,\bar{i} + xy \cdot \bar{j}$

$C : \bar{r}(t) = \cos(t)\,\bar{i} - \sin(t)\,\bar{j}$

$= (\cos(t), -\sin(t)) \quad 0 \leq t \leq \frac{\pi}{2}$

$$\int_C \bar{F} \cdot d\bar{r} = \int_0^{\frac{\pi}{2}} \bar{F}(r(t)) \cdot r'(t)\, dt$$

Vi trenger

$r'(t) = (-\sin(t), -\cos(t))$

$\bar{F}(r(t)) = (\cos^2(t), -\cos(t)\sin(t))$

$$= \int_0^{\frac{\pi}{2}} \left(\cos^2(t), -\cos(t)\sin(t)\right)\cdot\left(-\sin(t), -\cos(t)\right) dt$$

$$= \int_0^{\frac{\pi}{2}} -\cos^2(t)\sin(t) + \cos^2(t)\sin(t)\ dt$$

$$= \int_0^{\frac{\pi}{2}} 0\ dt = \underline{\underline{0}}$$

4. $\bar{F}(x,y,z) = \left(\frac{z}{x}, y, x\right)$

$C: \ \bar{r}(t) = (e^t, \ln t, \sin t) \qquad 1 \le t \le 2$

$\bar{r}'(t) = \left(e^t, \frac{1}{t}, \cos t\right)$

$$\int_C \bar{F}\cdot d\bar{r} = \int_1^2 \bar{F}(\bar{r}(t))\cdot \bar{r}'(t)\ dt$$

$$= \int_1^2 \left(\frac{\sin t}{e^t}, \ln t, e^t\right)\cdot\left(e^t, \frac{1}{t}, \cos t\right) dt$$

$$= \int_1^2 \left(\sin t + \frac{\ln t}{t} + e^t \cdot \cos t\right) dt$$

$$= \left[-\cos t + \frac{1}{2}(\ln t)^2 + \frac{1}{2}e^t(\cos t + \sin t)\right]_1^2$$

$$= -\cos 2 + \frac{1}{2}(\ln 2)^2 + \frac{1}{2}e^2(\cos 2 + \sin 2)$$

$$+\cos 1 - \tfrac{1}{2}(\ln 1)^2 - \tfrac{1}{2}e(\cos 1 + \sin 1)$$

$$\int e^t \cdot \cos t \, dt = e^t \cos t + \int e^t \sin t \, dt$$

$u' = e^t$, $v = \cos t$; $u = e^t$, $v' = -\sin t$

$u' = e^t$, $v = \sin t$; $u = e^t$, $v' = \cos t$

$$= e^t \cdot \cos t + e^t \sin t - \int e^t \cos t \, dt$$

$$\int e^t \cos t \, dt = \frac{1}{2} e^t \cos t + \frac{1}{2} e^t \sin t$$

5. $\bar{F} = (yz, x, xy)$ $\quad \bar{r}(t) = (t, \operatorname{arctg} t, t)$ $\quad$ (x, y, z)

$0 \le t \le 1$

$$\bar{r}'(t) = \left(1, \frac{1}{1+t^2}, 1\right)$$

$$\int_C \bar{F} \cdot d\bar{r}$$

$$= \int_0^1 (t \operatorname{arctg} t, t, t \operatorname{arctg} t) \cdot \left(1, \frac{1}{1+t^2}, 1\right) dt$$

$$= \int_0^1 t \operatorname{arctg} t + \frac{t}{1+t^2} + t \cdot \operatorname{arctg} t \, dt$$

$$= \int_0^1 2t \operatorname{arctg} t + \frac{t}{1+t^2} \, dt$$

$u = 1+t^2$

$$= \left[t^2 \operatorname{arctg} t\right]_0^1 - \int_0^1 dt + \int_0^1 \frac{1}{1+t^2}\,dt + \int_0^1 \frac{t}{1+t^2}\,dt$$

$$= \operatorname{arctg} 1 - 1 + \left(\operatorname{arctg} 1 - 0\right) + \left[\frac{1}{2}\ln(1+t^2)\right]_0^1$$

$$= 2\operatorname{arctg} 1 - 1 + \frac{1}{2}\ln 2 =$$

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$$= 2\cdot\frac{\pi}{4} - 1 + \frac{1}{2}\ln 2 = \frac{\pi}{2} - 1 + \frac{1}{2}\ln 2$$

$$\int 2t \operatorname{arctg} t\,dt = t^2 \operatorname{arctg} t - \int \frac{t^2}{1+t^2}\,dt \qquad \frac{t^2}{1+t^2} = \frac{1+t^2-1}{1+t^2} = 1 - \frac{1}{1+t^2}$$

$u' = 2t \quad v = \operatorname{arctg} t$

$u = t^2 \quad v' = \frac{1}{1+t^2}$

$$= t^2 \operatorname{arctg} t - \int 1\,dt + \int \frac{1}{1+t^2}\,dt$$

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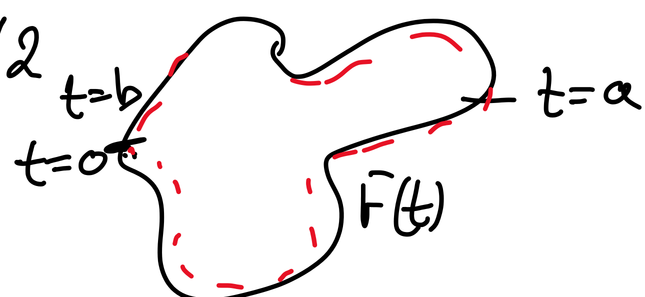

$\bar{F}$

1) $$\int_C \bar{F}\cdot d\bar{r} = \int_0^b \bar{F}(\bar{r}(t))\cdot \bar{r}'(t)\,dt$$

2) $$\int_{C'} \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t))\cdot \bar{r}'(t)\,dt$$

$$+ \int^{a} \bar{F}(\bar{r}(t))\cdot \bar{r}'(t)\,dt$$

$$\boxed{1) = 2)}$$

3.5 7. $\int_C \bar{F} \cdot d\bar{r}$ $\quad F(x,y) = (2xy, x^2)$

$C: \bar{r}(t) = (2t\cos t, \sin t) \quad 0 \leq t \leq \frac{\pi}{2}$

Bruker at $\bar{F}$ er konservativt:

Kan vise $\nabla\varphi = F$

hvor $\varphi(x,y) = x^2 y$

$$\int_C \bar{F} \cdot d\bar{r} = \varphi(\bar{r}(\tfrac{\pi}{2})) - \varphi(\bar{r}(0))$$

$$= \varphi(\pi \cdot \cos\tfrac{\pi}{2}, \sin\tfrac{\pi}{2}) - \varphi(0,0)$$

$$= \varphi(0,1) - \varphi(0,0) = \underline{\underline{0}}$$