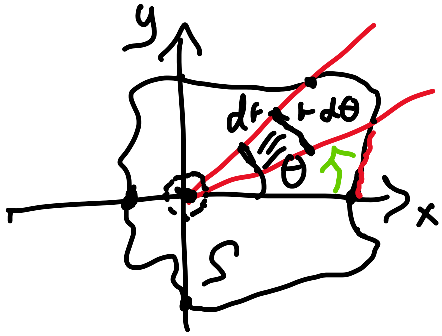
Forelesning, 2. mars 2021

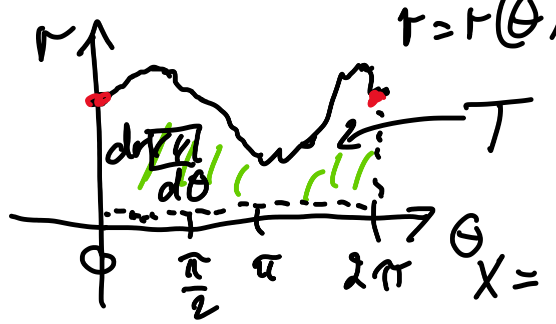
tirsdag 2. mars 2021 12:15

Dobbeltintegraler i polarkoordinater

Anvendelser av $\iint$: - areal
- massemiddelpunkt

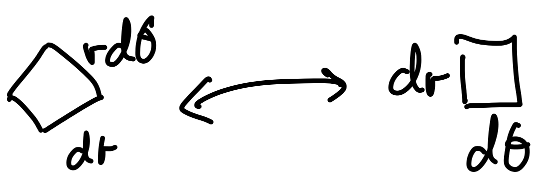
Flateintegraler : - areal
- skalarfunksjoner





$x = r \cdot \cos\theta$

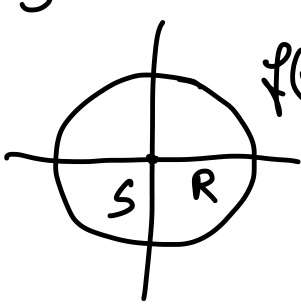
$y = r \sin\theta$



Funksjon $f = f(x,y)$

$$\iint_S f \, dx\, dy = \iint_T f(r\cos\theta, r\sin\theta) \cdot r \cdot dr\, d\theta$$

EKS



$f(x,y) = 1$

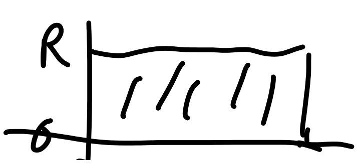
$$\text{areal}(S) = \iint_S 1 \, dS$$

$dy \boxed{dS}\, dx$

$dS = dx\, dy$
(forkortet skrivemåte)

$0 \le \theta \le 2\pi$

$0 \le r \le R$



$$\iint_S 1 \, dS = \int_0^{2\pi} \int_0^{R} 1 \cdot r \, dr\, d\theta$$

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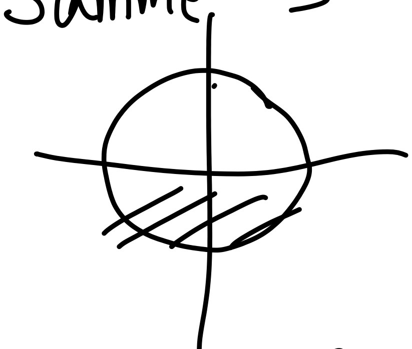
$$= \int_0^{2\pi} \left[\frac{1}{2} r^2\right]_0^R d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} R^2 \, d\theta = \frac{R^2}{2}\left[\theta\right]_0^{2\pi}$$

$$= \frac{R^2}{\cancel{2}} \cdot \cancel{2}\pi = \underline{\underline{\pi R^2}}$$

Samme S



$$\iint_S f \, dS = \int_0^{2\pi}\int_0^R r^3 \cos^2\theta \sin\theta \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^R r^4 \cos^2\theta \sin\theta \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{5} r^5 \cos^2\theta \sin\theta\right]_0^R d\theta$$

$$= \frac{1}{5} R^5 \int_0^{2\pi} \cos^2\theta \sin\theta \, d\theta$$

$f(x,y) = x^2 y$

$$\boxed{\begin{aligned} f(r\cos\theta, r\sin\theta) &= r^2\cos^2\theta \cdot r\sin\theta \\ &= r^3 \cos^2\theta \sin\theta \end{aligned}}$$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$$= \frac{1}{5} R^5 \left[-\frac{1}{3}\cos^3\theta\right]_0^{2\pi}$$

$$= \frac{1}{5} R^5 \left(-\frac{1}{3}\right) \cos^3(2\pi) - \cos^3(0) = 0$$

Integral over øvre halvsirkel. (alternativ)

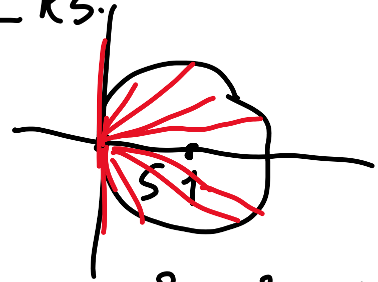
dvs $0 \le \theta \le \pi$

Da hadde vi fått $= \frac{1}{5} R^5 \left(-\frac{1}{3}\right)\left(\cos^3(\pi) - \cos^3(0)\right)$

$$\frac{1}{5} R^5 \left(-\frac{1}{3}\right)\left((-1)^3 - 1^3\right) = \frac{2}{\ } R^5$$

$= \frac{\cdot}{5}\pi \left(\cdots \frac{\cdot}{3} \right)\left(\cdots \right) = \underline{\underline{\frac{\cdots}{15}}}$

Eks.



$(x-1)^2 + y^2 \leq 1$

$\underline{x^2 - 2x + y^2 \leq 0}$

I polarkoordinater

$x = r\cos\theta \quad y = r\sin\theta$

$(r\cos\theta)^2 - 2(r\cos\theta) + (r\sin\theta)^2 \leq 0$

$\underbrace{r^2\cos^2\theta + r^2\sin^2\theta}_{r^2} - 2r\cos\theta \leq 0$

$r^2 - 2r\cos\theta \leq 0$

$r(r - 2\cos\theta) \leq 0 \qquad r \geq 0$

$\Longrightarrow \quad r - 2\cos\theta \leq 0$

dvs $\quad r \leq 2\cos\theta$

Området i (θ, r)-planet:

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$0 \leq r \leq 2\cos\theta$

Integrere

$f(x,y) = \sqrt{x^2+y^2}$

$f(r\cos\theta, r\sin\theta) = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}$

$= r$

$$\iint_S f \, dS = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r \cdot r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{3} r^3 \right]^{2\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}} \left[\frac{1}{3} r^3 \right]_0 d\theta$$

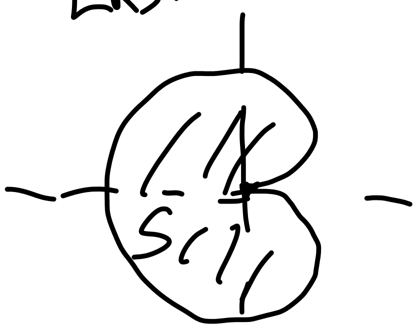
$$= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^3 \, d\theta$$

$$= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \, d\theta \qquad \cos^3\theta = \cos^2\theta \cdot \cos\theta = (1-\sin^2\theta)\cos\theta$$

$$= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta - \sin^2\theta\cos\theta \, d\theta = \underline{\underline{\frac{32}{9}}}$$

Litt regning

Eks.



$$\text{areal}(S) = \iint_S 1 \, dS$$

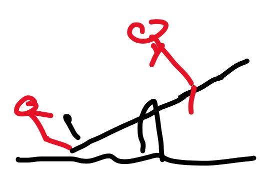
$$= \int_0^{2\pi} \int_0^{\sin\frac{\theta}{2}} 1 \cdot r \, dr \, d\theta = \underline{\underline{\frac{\pi}{2}}}$$

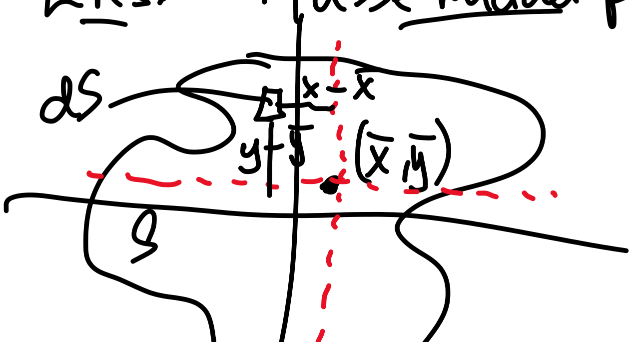
Litt regning

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq \sin\frac{\theta}{2}$

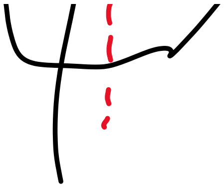
Eks. Masse middel punkt.





$$\iint_S (x - \bar{x}) \, dx\,dy = 0$$

$$\iint x \, dx\,dy - \iint \bar{x} \, dx\,dy = 0$$

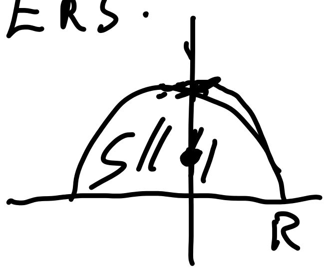


$$\iint_S x\,dx\,dy - \bar{x}\underbrace{\iint_S dx\,dy}_{\text{areal}(S)} = 0$$

$$\bar{x} = \frac{1}{\text{areal}(S)}\iint_S x\,dx\,dy$$

$$\bar{y} = \frac{1}{\text{areal}(S)}\iint_S y\,dx\,dy$$

Eks.



areal $\frac{1}{2}\pi R^2$

$0 \le \theta \le \pi$

$0 \le r \le R$

Symmetri: $\bar{x} = 0$ $\quad \bar{y} = ?$

$$\bar{y} = \frac{1}{\text{areal}(S)}\iint_S y\,dx\,dy \qquad y = r\sin\theta$$

$$\iint_S y\,dx\,dy = \int_0^\pi \int_0^R r\sin\theta \cdot r\,dr\,d\theta$$

$$= \frac{1}{3}R^3\int_0^\pi \sin\theta\,d\theta$$

$$= \frac{1}{3}R^3\left[-\cos\theta\right]_0^\pi$$

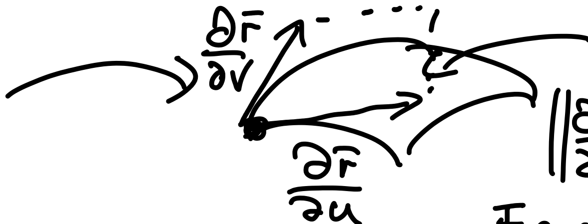
$$= \frac{2}{3}R^3$$

$$\bar{y} = \frac{1}{\frac{1}{2}\pi R^2}\cdot\frac{2}{3}R^3 = \underline{\underline{\frac{4}{3\pi}R}}$$

Areal av mer generelle flater

Parametrisering $\bar r(u,v) = \big(x(u,v),\, y(u,v),\, z(u,v)\big)$

$dv\,\boxed{dS}$
du

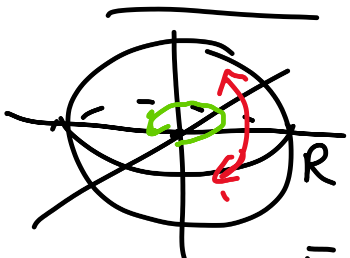


$\left\|\frac{\partial \bar r}{\partial u} \times \frac{\partial \bar r}{\partial v}\right\|$

Forstørrelsesfaktor mellom de to firkanter

$$dA = \left\| \frac{\partial \bar r}{\partial u} \times \frac{\partial \bar r}{\partial v} \right\| dS$$

Areal av kuleflate.



$$0 \le \theta \le 2\pi$$
$$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$$

$$\bar r(\theta,\varphi) = \Big(R\cos\theta\cdot\cos\varphi,\ R\sin\theta\cos\varphi,\ \overset{z}{\overset{\parallel}{R\sin\varphi}}\Big)$$

$$\frac{\partial \bar r}{\partial \theta} = (-R\sin\theta\cos\varphi,\ R\cos\theta\cos\varphi,\ 0)$$

$$\frac{\partial \bar r}{\partial \varphi} = (-R\cos\theta\sin\varphi,\ -R\sin\theta\sin\varphi,\ R\cos\varphi)$$

$$\frac{\partial \bar r}{\partial \theta} \times \frac{\partial \bar r}{\partial \varphi} = \begin{vmatrix} \bar i & \bar j & \bar k \\ -R\sin\theta\cos\varphi & R\cos\theta\cos\varphi & 0 \\ -R\cos\theta\sin\varphi & -R\sin\theta\sin\varphi & R\cos\varphi \end{vmatrix}$$

$$= \big(R^2\cos\theta\cos^2\varphi,\ R^2\sin\theta\cos^2\varphi,\ R^2\cos\varphi\sin\varphi\big)$$

$$\left\| \frac{\partial \bar r}{\partial \theta} \times \frac{\partial \bar r}{\partial \varphi} \right\| = R^2 \ldots$$

$$\left\|\frac{\partial F}{\partial\theta}\times\frac{\partial F}{\partial\varphi}\right\| = R^2\cos\varphi = J$$

$$\text{areal} = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\cdot R^2\cos\varphi \, d\varphi \, d\theta$$

kan erstattes av en funksjon $f(F(\varphi,\theta))$

$$= R^2\int_0^{2\pi}\left[\sin\varphi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta$$

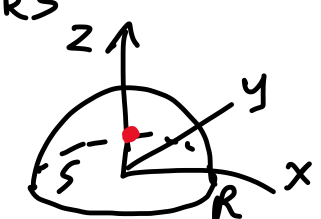
$$= R^2\int_0^{2\pi}\underbrace{1-(-1)}_{=2}\, d\theta = R^2\cdot 2\cdot(2\pi-0)$$

$$= \underline{\underline{4\pi R^2}}$$

Flateintegral av et skalarfelt

$$\iint_S f\, dS = \iint_A f(F(u,v))\left\|\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right\| du\, dv$$

Eks



Halv kuleskall

areal $= 2\pi R^2$

Tyngdepunkt:

Symmetri: $\bar{x} = \bar{y} = 0 \quad \bar{z} = ?$

$$\iint_S z\, dS = \int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}} R\sin\varphi \, R^2\cos\varphi \, d\varphi\right)d\theta$$

$$= \left[\left(\int_0^{\frac{\pi}{2}} R^3\sin\varphi\cos\varphi \, d\varphi\right)\theta\right]_0^{2\pi}$$

$$= 2\pi R^3\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi \, d\varphi$$

$u = \sin\varphi$

$du = \cos\varphi \, d\varphi$

$\varphi = \frac{\pi}{2} \Rightarrow u = 1$

$$= 2\pi R^3\int_0^1 u\, du$$

$\varphi = 0 \Rightarrow u = 0$

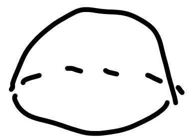
$$= 2\pi R^3 \left[\frac{1}{2}u^2\right]_0^1$$

$$= 2\pi R^3 \cdot \frac{1}{2} = \pi R^3$$

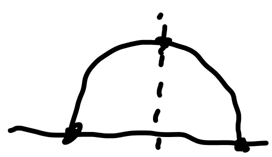
$$\Rightarrow \bar{z} = \frac{1}{2\cancel{\pi}\cancel{R^2}} \cdot \cancel{\pi} R^{\cancel{3}} = \frac{R}{2}$$

Digresjon:

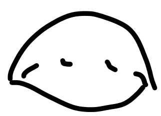
R = 1.



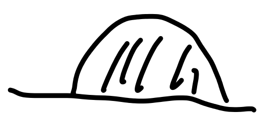
$\frac{1}{2}$ Kuleskall : $\frac{1}{2} = 0{,}5$



$\frac{1}{2}$ sirkel : $\frac{2}{\pi} \approx 0{,}64$

Massiv 

$\frac{1}{2}$ Kule : $\frac{3}{8} = 0{,}375$



$\frac{1}{2}$ Disk : $\frac{4}{3\pi} \approx 0{,}43$