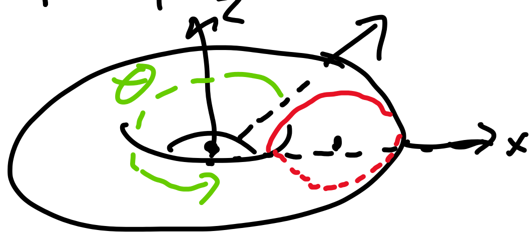
Forelesning, 3. mars 2021

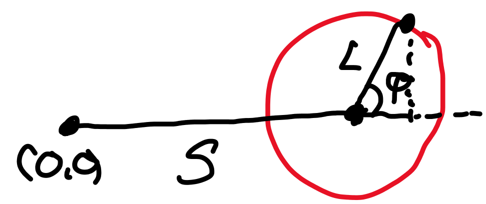
onsdag 3. mars 2021 12:11

Dobbeltintegraler

• skifte av variable.

Repetisjon





Torus: (Smultringflate)

$x = S + L\cdot\cos\varphi$

$z = L\cdot\sin\varphi$

$$\bar{r}(\theta,\varphi) = \big((S+L\cdot\cos\varphi)\cos\theta,\ (S+L\cos\varphi)\sin\theta,\ L\cdot\sin\varphi\big)$$

areal: integrere funksjonen $f(x,y,z) = 1$.

$$\frac{\partial\bar{r}}{\partial\theta} = \big(-(S+L\cos\varphi)\sin\theta,\ (S+L\cos\varphi)\cos\theta,\ 0\big)$$

$$= (S+L\cos\varphi)\,(-\sin\theta,\ \cos\theta,\ 0)$$

$$\frac{\partial\bar{r}}{\partial\varphi} = (-L\sin\varphi\cdot\cos\theta,\ -L\sin\varphi\sin\theta,\ L\cdot\cos\varphi)$$

$$= L\,(-\sin\varphi\cos\theta,\ -\sin\varphi\sin\theta,\ \cos\varphi)$$

$$\frac{\partial\bar{r}}{\partial\theta}\times\frac{\partial\bar{r}}{\partial\varphi} = (S+L\cos\varphi)\cdot L\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\sin\theta & \cos\theta & 0 \\ -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & \cos\varphi \end{vmatrix}$$

$$= (S + L\cos\varphi) L \left(\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin^2\theta\sin\varphi + \underbrace{\cos^2\theta\sin\varphi}_{\sin\varphi}\right)$$

$$\left\| \frac{\partial F}{\partial \theta} \times \frac{\partial F}{\partial \varphi} \right\| = \underline{\underline{(S + L\cos\varphi) L}}$$

$$\text{areal} = \int_0^{2\pi}\int_0^{2\pi} (S + L\cos\varphi) L \cdot d\varphi\, d\theta = \underline{\underline{4\pi^2 \cdot SL}}$$

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Husk: ($\mathbb{R}$)

substitusjon: $\int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta \, d\theta$

$u(\theta) = u = \sin\theta$

$du = \cos\theta \, d\theta$

$u_1 = \sin(\frac{\pi}{2}) = 1$

$u_2 = \sin(0) = 0$

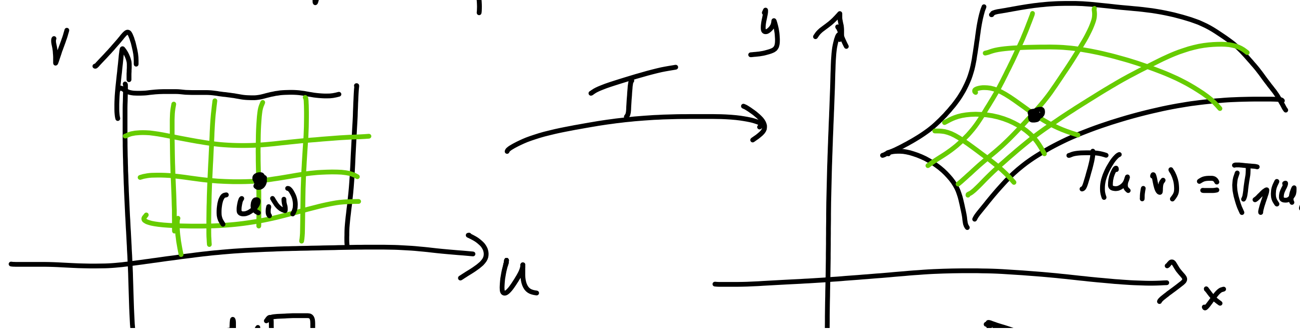
variabelskifte

$$= \int_0^{\frac{\pi}{2}} u(\theta)^2 \cdot u'(\theta) \cdot d\theta$$

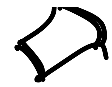
$$= \int_{u(0)}^{u(\frac{\pi}{2})} u^2 \, du = \frac{1}{3}$$

— o — o —

Variabelskifte i planet



$T(u,v) = (T_1(u,v), T_2(u,v))$

| $\frac{av|}{du}$ |  $\mathbb{R}^2 \subseteq \mathbb{R}^3$

$$F(u,v) = \left(T_1(u,v),\ T_2(u,v),\ 0\right)$$

$$\frac{\partial F}{\partial u} = \left(\frac{\partial T_1}{\partial u},\ \frac{\partial T_2}{\partial u},\ 0\right)$$

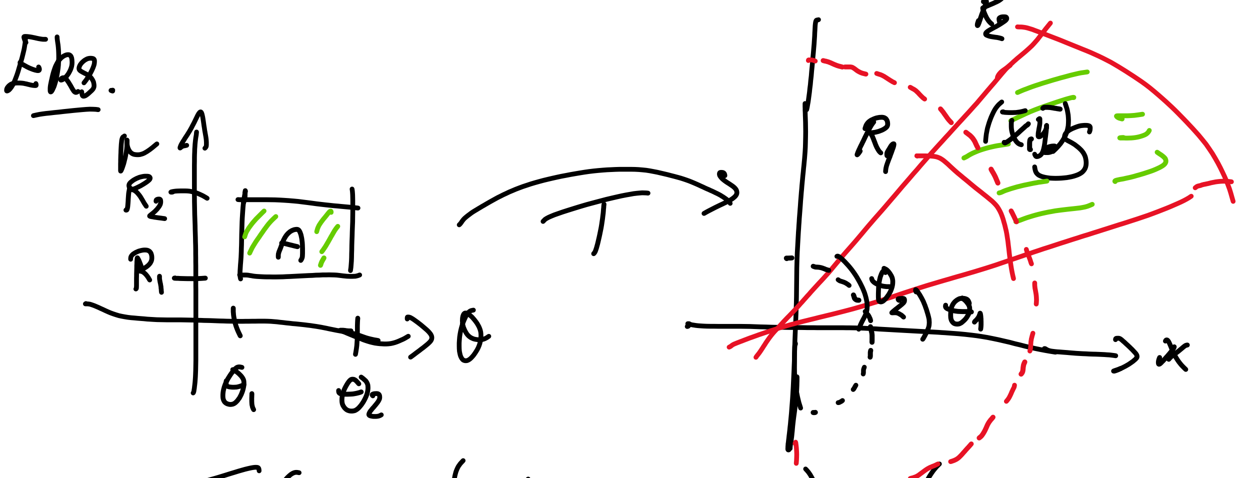
$$\frac{\partial F}{\partial v} = \left(\frac{\partial T_1}{\partial v},\ \frac{\partial T_2}{\partial v},\ 0\right)$$

$$\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} & 0 \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} & 0 \end{vmatrix} = \left(0, 0, \begin{vmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_2}{\partial u} \\ \frac{\partial T_1}{\partial v} & \frac{\partial T_2}{\partial v} \end{vmatrix}\right)$$

Transponerte har samme determinant

$$\text{Jacobi-determinanten} = \begin{vmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_1}{\partial v} \\ \frac{\partial T_2}{\partial u} & \frac{\partial T_2}{\partial v} \end{vmatrix}$$

Eks.



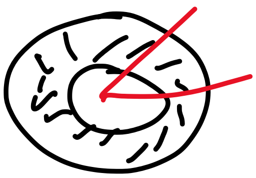
$$T(r,\theta) = \left(\underset{x}{T_1(r,\theta)},\ \underset{y}{T_2(r,\theta)}\right) = (r\cos\theta,\ r\sin\theta)$$

$$\begin{vmatrix} \frac{\partial T}{} & \frac{\partial T}{} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \end{vmatrix}$$

$$\text{Jacobi-determinant} = \begin{vmatrix} \frac{\partial T_1}{\partial r} & \frac{\partial T_1}{\partial \theta} \\ \frac{\partial T_2}{\partial r} & \frac{\partial T_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = \underline{\underline{r}}.$$

$$\text{areal} = \left(\pi R_2^2 - \pi R_1^2\right)\frac{\theta_2-\theta_1}{2\pi} = \frac{(R_2^2-R_1^2)(\theta_2-\theta_1)}{2}$$



$$\int_{\theta_1}^{\theta_2}\int_{R_1}^{R_2} \underbrace{r\cos\theta}_{= x} \cdot r \, dr\, d\theta$$

$$= \frac{1}{3}\left(R_2^3 - R_1^3\right)\left(\sin\theta_2 - \sin\theta_1\right)$$

$$\bar{x} = \frac{\frac{1}{3}\cancel{(R_2-R_1)}\left(R_2^2+R_1R_2+R_1^2\right)(\sin\theta_2-\sin\theta_1)}{\frac{1}{2}\cancel{(R_2-R_1)}(R_2+R_1)(\theta_2-\theta_1)}$$

$$= \frac{2}{3}\,\frac{R_2^2+R_1R_2+R_1^2}{R_2+R_1}\,\frac{\sin\theta_2-\sin\theta_1}{\theta_2-\theta_1}$$

Sett $R_1 = R_2$:

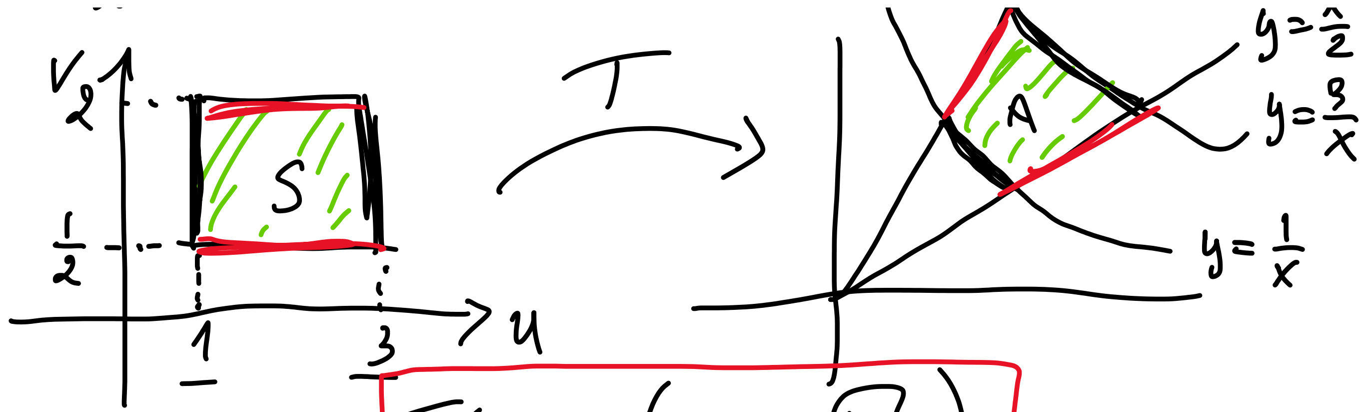
Sett $\theta_1 = -\frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$

$$\bar{X} = \frac{2}{3}\,\frac{R^2+R^2+R^2}{R+R}\,\frac{\sin\left(+\frac{\pi}{2}\right)-\sin\left(-\frac{\pi}{2}\right)}{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}$$

$$= \frac{\cancel{2}}{\cancel{3}}\,\frac{\cancel{3}R^{\cancel{2}}}{\cancel{2}\cancel{R}}\,\frac{1-(-1)}{\frac{\pi}{2}+\frac{\pi}{2}} = \frac{2R}{\pi} = \underline{\underline{\frac{2}{\pi}\cdot R}}$$

Eks.

$y = 2x$



$$\boxed{T(u,v) = \left(\sqrt{u\cdot v}, \sqrt{\frac{u}{v}}\right)}$$

$$\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{x} \qquad\quad \overset{\shortparallel}{y}$$

$$\left.\begin{aligned} x &= \sqrt{uv} \\ y &= \sqrt{\frac{u}{v}} \end{aligned}\right\} \Rightarrow \begin{aligned} u &= x\cdot y = \sqrt{u}\cdot\cancel{\sqrt{v}}\cdot\frac{\sqrt{u}}{\cancel{\sqrt{v}}} \\ v &= \frac{x}{y} = \frac{\cancel{\sqrt{u}}\sqrt{v}}{\frac{\cancel{\sqrt{u}}}{\sqrt{v}}} = \sqrt{v}\cdot\sqrt{v} \end{aligned}$$

S: $u = 1$ dvs $xy = 1$ eller $y = \frac{1}{x}$

$u = 3$ dvs $xy = 3$ eller $y = \frac{3}{x}$

$v = \frac{1}{2}$ dvs $\frac{x}{y} = \frac{1}{2}$ eller $y = 2x$

$v = 2$ dvs $\frac{x}{y} = 2$ eller $y = \frac{1}{2}x$

skal integrere $\frac{x}{y}$ over A: $\iint\limits_A \frac{x}{y}\,dx\,dy$

Jacobi-determinant: $\begin{vmatrix} \frac{\partial\sqrt{uv}}{\partial u} & \frac{\partial\sqrt{uv}}{\partial v} \\ \frac{\partial\sqrt{\frac{u}{v}}}{\partial u} & \frac{\partial\sqrt{\frac{u}{v}}}{\partial v} \end{vmatrix}$ $\qquad \sqrt{uv} = \sqrt{u}\cdot\sqrt{v}$, $\quad \sqrt{\frac{u}{v}} = \frac{\sqrt{u}}{\sqrt{v}}$

$= \begin{vmatrix} \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \\ \end{vmatrix} = \frac{\cancel{\sqrt{v}}}{\ldots}\left(-\frac{1}{\ldots}\right)\frac{\cancel{\sqrt{u}}}{\ldots} - \frac{\cancel{\sqrt{u}}}{\ldots}\cdot\frac{1}{\ldots}$

$$\left| \begin{matrix} \frac{1}{2\sqrt{u}\sqrt{v}} & -\frac{1}{2}\frac{\sqrt{u}}{(\sqrt{v})^3} \end{matrix} \right|$$

$$= -\frac{1}{4}\frac{1}{(\sqrt{v})^2} - \frac{1}{4}\frac{1}{(\sqrt{v})^2} = -\frac{1}{2v}$$

Absoluttverdi $= \frac{1}{2v}$

$$\iint_A \frac{x}{y}\,dx\,dy = \iint_S \frac{\sqrt{\cancel{u}v}}{\frac{\cancel{\sqrt{u}}}{\sqrt{v}}}\frac{1}{2v}\,du\,dv = \iint_S \frac{\sqrt{v}}{\frac{1}{\sqrt{v}}}\frac{1}{2v}\,du\,dv$$

$$= \int_{\frac{1}{2}}^{2}\int_{1}^{3} \frac{1}{2}\,du\,dv = \frac{1}{2}\left(2-\frac{1}{2}\right)(3-1) = \underline{\underline{\frac{3}{2}}}$$

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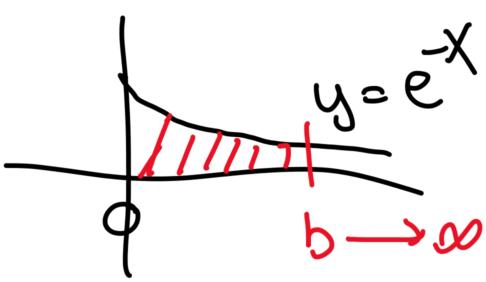
Uegentlige integraler

Eksempel fra 1-variabel-teori:



$$\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = 1$$

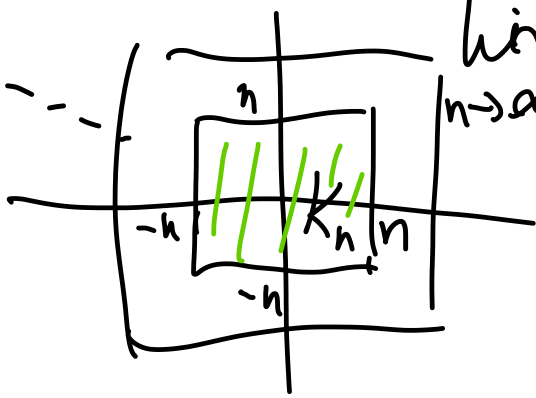
I planet?

$$f(x,y) = e^{-\frac{x^2+y^2}{2}}$$ over hele $\mathbb{R}^2$ eller $A \subseteq \mathbb{R}^2$

Hva betyr $\int_{}^{\infty}\int_{}^{\infty}$?

Mulighet 1:

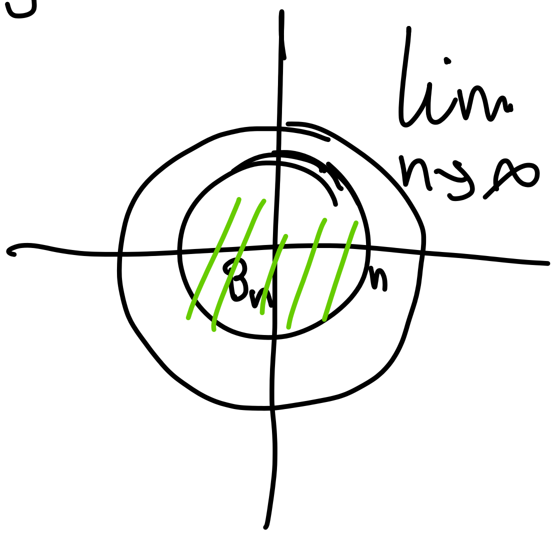
$-\infty \; -\infty$



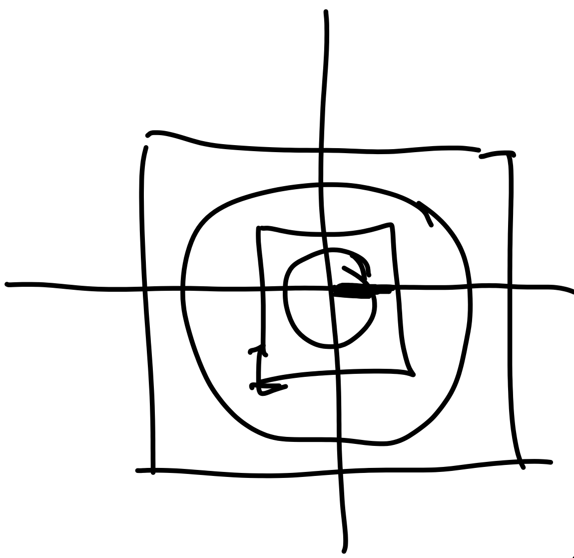
$$\lim_{n\to\infty} \iint_{A\cap K_n} f \, dx\,dy$$

$\|$ samme verdi

Mulighet 2



$$\lim_{n\to\infty} \iint_{A\cap B_n} f \, dx\,dy$$



f positiv funksjon

$B_n \subset K_n \subset B_{2n}$

$$\iint_{B_n} f \, dA \le \iint_{K_n} f \, dA$$

$$\le \iint_{B_{2n}} f \, dA$$

Grense

Ved squeezing principle :

$$\lim_{n\to\infty} \iint_{B_n} f\, dA = \lim_{n\to\infty} \iint_{K_n} f\, dA.$$

$$\iint_{\mathbb{R}^n} e^{-\frac{x^2+y^2}{2}}\, dx\, dy = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} \cdot r\, dr\, d\theta$$

$0 \le \theta \le 2\pi$

$0 \le r$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$$= 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} \cdot r\, dr.$$

$u = \frac{r^2}{2}$

$du = r\, dr$

$r=0 \quad u=0$

$r=b \quad u=\frac{b^2}{2}$

$$= 2\pi \lim_{b\to\infty} \int_0^{b} e^{-\frac{r^2}{2}} r\, dr$$

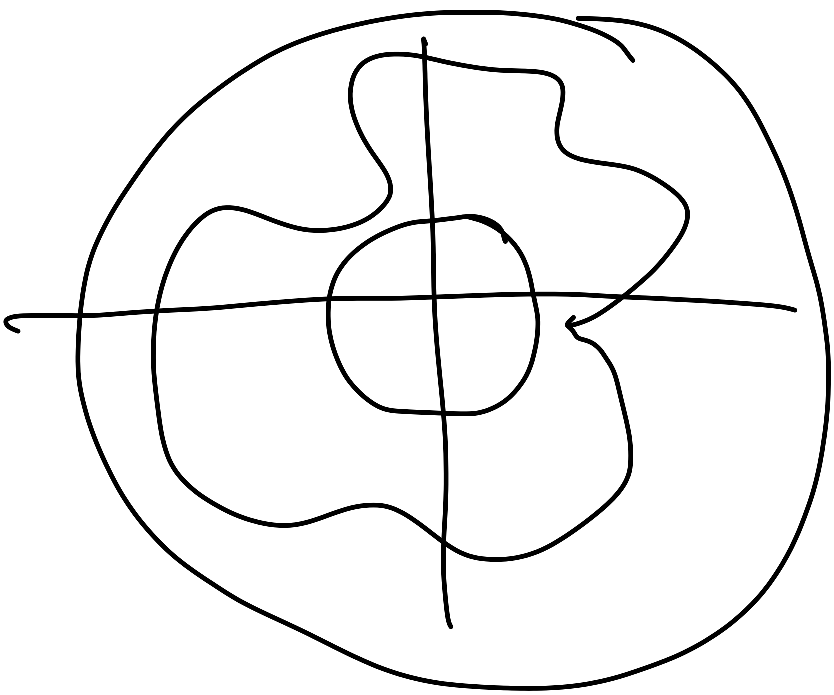
$$= 2\pi \lim_{b\to\infty} \int_0^{\frac{b^2}{2}} e^{-u}\, du$$

$$= 2\pi \lim_{b\to\infty} \left[-e^{-u}\right]_0^{\frac{b^2}{2}}$$

$$= 2\pi \lim_{b\to\infty} \left(-e^{-\frac{b^2}{2}} + 1\right) = \underline{\underline{2\pi}}$$

Hvis vi har funksjoner som ikke bare er positive :

$$f = \underset{\text{positiv}}{f_+} + \underset{\text{negativ}}{f_-}$$



- Kvadrat
- Disk.