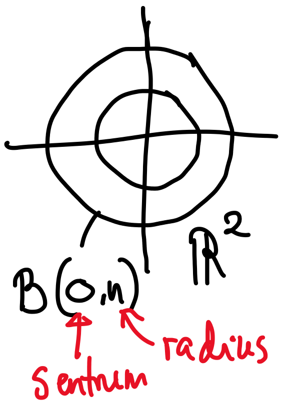
Forelesning, 8. mars 2021

mandag 8. mars 2021 14:17

* Repetisjon, uegentlige integraler
* Trippelintegraler
  - over rektangulære bokser
  - over generelle områder
* Anvendelser

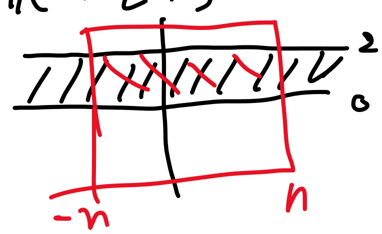


$f(x,y)$ positiv funksjon $\quad f = f_+ + f_-$

$$\iint_{\mathbb{R}^2} f\,dx\,dy = \lim_{n\to\infty} \iint_{B(0,n)} f\,dx\,dy$$

$B(0,n) \leftrightarrow K_n = [-n,n]\times[-n,n]$

Eks 1. $\quad f(x,y) = \dfrac{y^2}{1+x^2} \qquad S = \mathbb{R} \times [0,2]$



$$\iint_S \frac{y^2}{1+x^2}\,dx\,dy = \lim_{n\to\infty} \iint_{S\cap K_n} \frac{y^2}{1+x^2}\,dx\,dy$$

$$= \lim_{n\to\infty} \int_{-n}^{n}\int_0^2 \frac{y^2}{1+x^2}\,dy\,dx$$

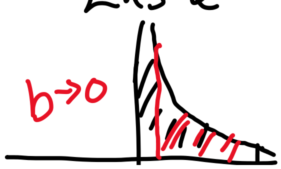
$$= \lim_{n\to\infty} \int_{-n}^{n} \left[\frac{1}{3}\,\frac{y^3}{1+x^2}\right]_0^2 dx$$

$$= \lim_{n\to\infty} \frac{8}{3}\int_{-n}^{n} \frac{dx}{1+x^2} = \lim_{n\to\infty} \frac{8}{3}\left(\operatorname{arctg}(n) - \operatorname{arctg}(-n)\right)$$

$$= \frac{8}{3}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \underline{\underline{\frac{8\pi}{3}}}$$

Eks 2



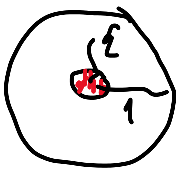
$f(x,y) = \dfrac{1}{(x^2+y^2)^p} \qquad p > 0$

$\int_a^b$ uegentlig integral

Integralet over $B(0,1)$

Problem: $(x,y) \to (0,0)$ så vil $f(x,y) \to \infty$

Integrerer f over $B(0,1) \setminus B(0,\varepsilon) = S_\varepsilon$



$$\iint_{S_\varepsilon} \frac{1}{(x^2+y^2)^p} \, dx \, dy$$

Polarkoordinater

$x = r\cos\theta$

$y = r\sin\theta$

$0 \leq \theta \leq 2\pi$

$\varepsilon \leq r \leq 1$

$x^2 + y^2 = r^2$

$$= \int_\varepsilon^1 \int_0^{2\pi} \frac{1}{r^{2p}} \, r \, d\theta \, dr$$

$$= \int_\varepsilon^1 \int_0^{2\pi} r^{1-2p} \, d\theta \, dr$$

$$= 2\pi \int_\varepsilon^1 r^{1-2p} \, dr = 2\pi \left[\frac{1}{2-2p} r^{1-2p+1} \right]_\varepsilon^1$$

$$= \frac{2\pi}{2-2p} \left(1 - \varepsilon^{2-2p}\right) = \underline{\frac{\pi}{1-p}\left(1 - \varepsilon^{2-2p}\right)}$$

$$\iint_{B(0,1)} f \, dx dy = \lim_{\varepsilon \to 0} \frac{\pi}{1-p}\left(1 - \varepsilon^{2-2p}\right) = \underline{\underline{\frac{\pi}{1-p}}} \qquad 0 < p < 1 \quad 0 < 2-2p < 2$$

Trippelintegral

$$f(x,y,z) = x + 2yz$$

$$R = [0,1] \times [1,2] \times [-1,1]$$

$$\iiint_R f \, dx dy dz = \int_0^1 \int_1^2 \int_{-1}^1 x + 2yz \, dz \, dy \, dx$$

$$= \int \int \left[xz + yz^2 \right]^1 dy \, dx$$

$$\int_0^1 \int_1^2 \left[xz + yz^2\right]_{-1}^{1} dy\,dx$$

$$= \int_0^1 \int_1^2 x + \cancel{y} - (-x) \cancel{-y}\; dy\,dx$$

$$= \int_0^1 \int_1^2 2x\; dy\,dx = \underline{\underline{1}}$$

MATLAB

```
triplequad(@(x,y,z)(x+2*y.*z),0,1,1,2,-1,1)
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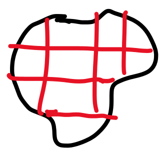
Vanskelighetene:

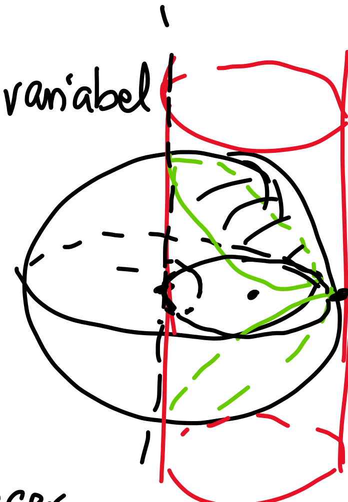
1-variabel:

2-variabel:

Type I / II



3-variabel

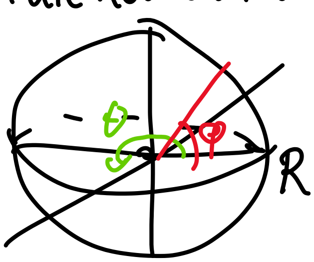


sentrum: 0
radius: 2

Den delen av kula som ligger inni sylinderen

$$\begin{cases} x^2 + y^2 + z^2 \le 4 \\ (x-1)^2 + y^2 \le 1 \end{cases}$$

Kulekoordinater



$$0 \le \theta \le 2\pi$$
$$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$$
$$0 \le r \le R$$

$$x = r\cos\theta\cos\varphi$$
$$y = r\sin\theta\cos\varphi$$
$$z = r\sin\varphi$$
$$x^2+y^2+z^2 = r^2$$

Kula: $0 \le r \le 2$

Sylinder:

$$(r\cos\theta\cos\varphi - 1)^2 + (r\sin\theta\cos\varphi)^2 \le 1$$

$$r^2\cos^2\theta\cos^2\varphi - 2r\cos\theta\cos\varphi + \cancel{1} + r^2\sin^2\theta\cos^2\varphi \le \cancel{1}$$

$$r^2\cos^2\varphi - 2r\cos\theta\cos\varphi \le 0$$

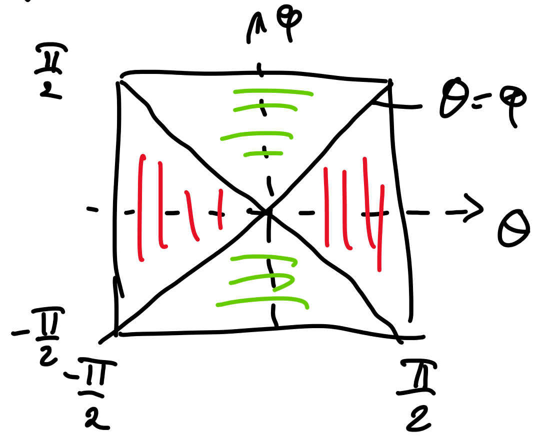
$$\underbrace{r}_{\ge 0}\,\underbrace{\cos\varphi}_{\ge 0}\left(r\cos\varphi - 2\cos\theta\right) \le 0$$

$$r\cos\varphi \le 2\cos\theta \quad \leftarrow \quad \theta=\varphi \quad r\le 2$$

Hvis likhet

$$r = \frac{2\cos\theta}{\cos\varphi}$$

To deler:



$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $\qquad D \subset \mathbb{R}^3$ lukket begrenset

- injektiv (1-1)
- kontinuerlige partiellderiverte
- $\det(T') \neq 0$

$$\iiint_{T(D)} f(x,y,z)\,dx\,dy\,dz = \iiint_{D} f(T(u,v,w))\,\left|\det(T'(u,v,w))\right|\,du\,dv\,dw$$

hvor
$$T' = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

Kule koordinater

$$T = T(\theta,\varphi,r)$$

$$x = r\cos\theta\cos\varphi$$
$$y = r\sin\theta\cos\varphi$$
$$z = r\sin\varphi$$

$$\det(T') = \begin{vmatrix} -r\sin\theta\cos\varphi & -r\cos\theta\sin\varphi & \cos\theta\cos\varphi \\ r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi & \sin\theta\cdot\cos\varphi \\ 0 & r\cos\varphi & \sin\varphi \end{vmatrix}$$

$$= r^2\cdot\cos\varphi$$

Volum av en kule med radius $= R$.

Kulekoordinater

$$D: \begin{aligned} 0 \leq \theta \leq 2\pi \\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \\ 0 \leq r \leq R \end{aligned}$$

Volum er integral av funksjonen $f(x,y,z) = 1$.

$$\iiint_{\text{Kule}} 1 \, dx\,dy\,dz = \iiint_D 1\cdot r^2\cdot\cos\varphi \, d\theta\,d\varphi\,dr$$

$$= \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^R r^2\cos\varphi \, dr\,d\varphi\,d\theta$$

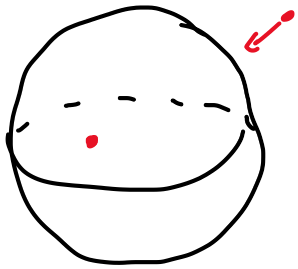
$$= \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{3}r^3\cdot\cos\varphi\right]_0^R d\varphi\,d\theta$$

$$= \frac{1}{3}R^3 \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \, d\varphi\,d\theta$$

$$= \frac{1}{3}R^3 \int_0^{2\pi} \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) d\theta$$

$$= \frac{1}{3}R^3 \int_0^{2\pi} (1-(-1))\,d\theta$$

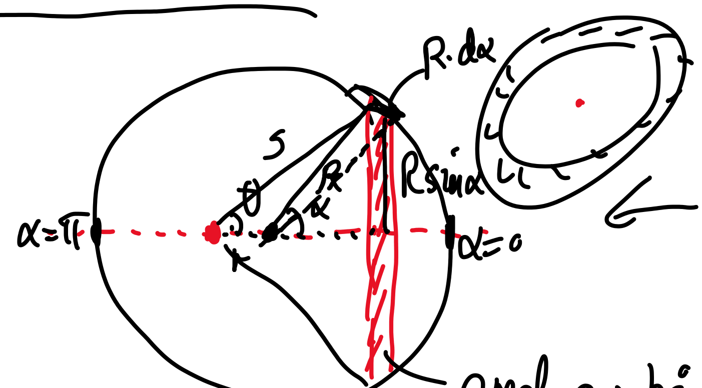
$$= \frac{1}{3}R^3 \cdot 2\pi \cdot 2 = \underline{\underline{\frac{4}{3}\pi R^3}}$$



Kuleskall

Gravitasjon $F \sim \frac{1}{r^2}$

(omvendt proposjonal med kvadratet av avstanden)



areal av bånd

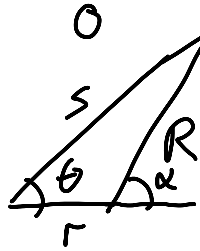
$2\pi R \sin\alpha \cdot R\,d\alpha$

$= 2\pi R^2 \sin\alpha \cdot d\alpha$

avstand: s

Tiltrekning: $\frac{2\pi R^2 \sin\alpha\, d\alpha}{s^2} \cdot \cos\theta$

$$I = 2\pi R^2 \int_0^{\pi} \frac{\sin\alpha \cdot \cos\theta\, d\alpha}{s^2}$$



Cosinus-setningen

$R^2 = s^2 + r^2 - 2sr\cos\theta \Rightarrow \cos\theta = \frac{s^2 + r^2 - R^2}{2sr}$

$s^2 = r^2 + R^2 - 2rR \cdot \cos(\pi - \alpha)$

$= r^2 + R^2 + 2rR\cos\alpha$

Den verer på begge sider

$2s\,ds = -2rR\sin\alpha\, d\alpha$

$\alpha = 0$: $s^2 = r^2 + R^2 + 2rR \Rightarrow s = r + R$

$\alpha = \pi$: $s = R - r$

$$I = 2\pi R^2 \int_{R+r}^{R-r} \frac{1}{s^2} \frac{2s\,ds}{-2rR} \frac{s^2 + r^2 - R^2}{2sr}$$

$$= \cancel{2}\pi R^{\cancel{2}} \int_{R-r}^{R+r} \frac{\cancel{2}\cancel{s}(s^2 + r^2 - R^2)}{s^2 \cdot \cancel{-2}r\cancel{R}\,\cancel{2s}r} ds = \frac{\pi R}{r^2} \int \frac{s^2 + r^2 - R^2}{s^2} ds$$

$R+r$

$R-r$

$= 0$