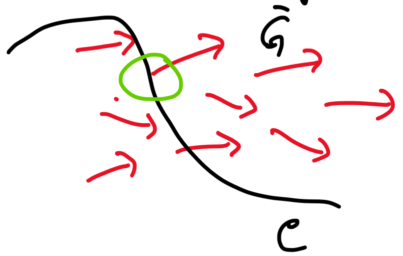
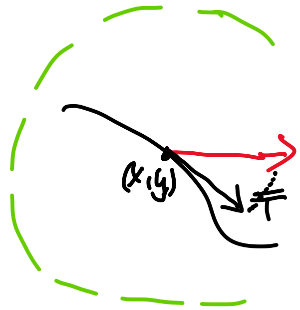
Forelesning, 9. mars 2021

tirsdag 9. mars 2021 12:14

Flateintegraler av vektorfelt (Greens teorem)

Motivasjon: Linjeintegral av vektorfelt





$\bar{r}(t)$: kurven

$(x,y) = \bar{r}(t_0)$

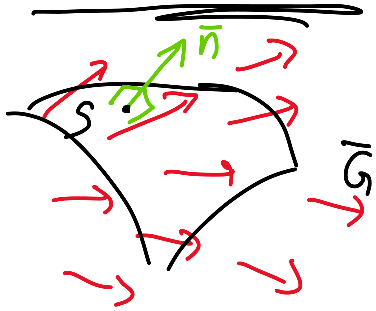
$\bar{T} = \dfrac{\bar{r}'(t)}{\|\bar{r}'(t)\|}$

$a \leq t \leq b$

$$\int_C \bar{G} \cdot d\bar{r} = \int_a^b \bar{G}(\bar{r}(t)) \cdot \bar{T} \, \|\bar{r}'(t)\| \, dt$$

$$= \int_a^b G(\bar{r}(t)) \cdot \bar{r}'(t) \, dt$$

Går fra kurve til flate.



Vektorfelt i 3 dimensjoner

Flate S: $\bar{r}(u,v)$, felt: $\bar{G}$

$\bar{n}$: enhets flatenormal

$$\iint_S \bar{G} \cdot \bar{n} \, dS$$

Flateintegralet av $\bar{G}$ over S.

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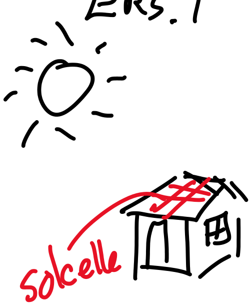
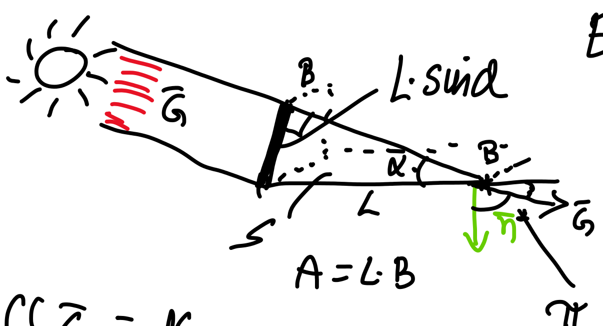
Fluksen av $\bar{G}$ gjennom S

Hvorfor?

Eks. 1





Energimengden

$\|\bar{G}\| \cdot L \cdot \sin\alpha \cdot B$

$= \|\bar{G}\| \cdot A \cdot \sin\alpha$

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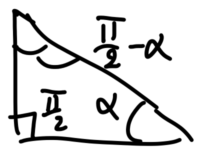
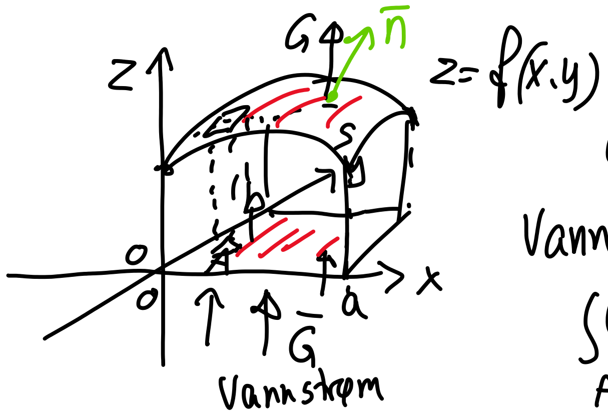
$\iint \bar{G} \cdot \bar{n} \, dS$



$$\bar{G}\cdot\bar{n} = \|\bar{G}\|\cdot\underbrace{\|\bar{n}\|}_{1}\cdot\underbrace{\cos\left(\tfrac{\pi}{2}-\alpha\right)}_{\sin\alpha}$$

$$\iint_S \bar{G}\cdot\bar{n}\,dS = \|G\|\cdot\sin\alpha\cdot A$$

Eks. 2



$$\bar{G} = g(x,y)\,\bar{k} \quad (\text{FeH})$$

Vann gjennom A:

$$\underline{\underline{\iint_A g(x,y)\,dx\,dy}}$$

$=$ Vann gjennom S.

Parametriser S:

$$\bar{r}(x,y) = (x, y, f(x,y))$$

$$\frac{\partial\bar{r}}{\partial x} = \left(1, 0, \frac{\partial f}{\partial x}\right)$$

$$\frac{\partial\bar{r}}{\partial y} = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial\bar{r}}{\partial x}\times\frac{\partial\bar{r}}{\partial y} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y}\end{vmatrix}$$

$$= \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = \bar{N}$$

$$\Rightarrow \quad \bar{n} = \frac{\bar{N}}{\|\bar{N}\|}$$

$$\bar{G}\cdot\bar{n} = g(x,y)\bar{k}\cdot\frac{1}{\|\bar{N}\|}\left(\underbrace{-\frac{\partial f}{\partial x}}_{\bar{i}}, \underbrace{-\frac{\partial f}{\partial y}}_{\bar{j}}, \underbrace{1}_{\bar{k}}\right) = \frac{1}{\|\bar{N}\|}\,g(x,y)$$

$$\iint_S \bar{G}\cdot\bar{n} = \iint_A \bar{G}(\bar{r}(x,y))\cdot\bar{n}\,\|\bar{N}\|\,dx\,dy$$

$$= \iint_A \frac{g(x,y)}{\|\bar{N}\|}\cdot\|\bar{N}\|\,dx\,dy = \underline{\underline{\iint_A g(x,y)\,dx\,dy}}$$

$\bar{r}: A \to \mathbb{R}^3$

$$\iint \bar{G}\cdot\bar{n}\,dS = \iint \bar{G}(\bar{r}(u,v))\left(\frac{\partial\bar{r}}{\partial u}\times\frac{\partial\bar{r}}{\partial v}\right)du\,dv$$

$\bar{F}(A) = S \qquad \iint_S \bar{G}\cdot\bar{n}\, dS = \iint_A \bar{G}(\bar{F}(u,v))\cdot\left(\frac{\partial \bar{F}}{\partial u} \times \frac{\partial \bar{F}}{\partial v}\right) \ldots$

flate

$\bar{G}$: felt

Eks

$$\bar{G}(x,y,z) = (2, x, yz) \qquad \frac{\partial \bar{F}}{\partial u} = (1, 1, v)$$

$$\bar{F}(u,v) = (u+v, u-v, u\cdot v) \qquad \frac{\partial \bar{F}}{\partial v} = (1, -1, u)$$

$0 \le u,v \le 1$

$$\frac{\partial \bar{F}}{\partial u} \times \frac{\partial \bar{F}}{\partial v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & v \\ 1 & -1 & u \end{vmatrix} = (u+v, v-u, -2)$$

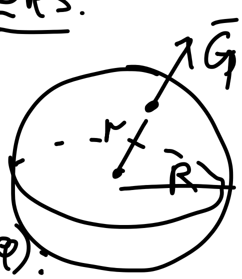
$$\bar{G}(\bar{F}(u,v)) = (2, u+v, (u-v)\cdot u\cdot v)$$

$$\iint_S \bar{G}\cdot\bar{n}\, dS = \int_0^1 \int_0^1 (2, u+v, (u-v)uv) \cdot (u+v, v-u, -2)\, du\, dv$$

$$= \int_0^1 \int_0^1 2u + 2v + v^2 - u^2 - 2u^2 v + 2uv^2 \, du\, dv$$

$$= \underline{\underline{2}}$$

Eks.



Gravitasjon

$R^2 = x^2 + y^2 + z^2$

$$\bar{G} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{R^2}$$

$$= \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$$

$\bar{F}(\theta,\varphi)$:

$x = R\cdot\cos\theta\cos\varphi$

$y = R\sin\theta\cos\varphi$

$z = R\cdot\sin\varphi$

$0 \le \theta \le 2\pi$

$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

$$G(\bar{F}(\theta,\varphi)) = \frac{1}{R^3}(R\cos\theta\cos\varphi, R\sin\theta\cos\varphi, R\sin\varphi)$$

$$= \frac{1}{R^2}(\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi)$$

$$\frac{\partial \bar{F}}{\partial \theta} = (-R\sin\theta\cos\varphi, R\cos\theta\cos\varphi, 0)$$

$$\frac{\partial \bar r}{\partial \varphi} = (-R\cos\theta\sin\varphi, -R\sin\theta\sin\varphi, R\cos\varphi)$$

$$\frac{\partial \bar r}{\partial \theta} \times \frac{\partial \bar r}{\partial \varphi} = \left(R^2\cos\theta\cos^2\varphi,\ R^2\sin\theta\cos^2\varphi,\ R^2\cos\varphi\sin\varphi\right)$$

Litt regning

$$\bar G(\bar r(\theta,\varphi))\cdot\left(\frac{\partial \bar r}{\partial \theta}\times\frac{\partial \bar r}{\partial \varphi}\right) = \cos^2\theta\cos^3\varphi + \sin^2\theta\cos^3\varphi + \cos\varphi\sin^2\varphi$$

$$= \cos^3\varphi + \cos\varphi\sin^2\varphi$$

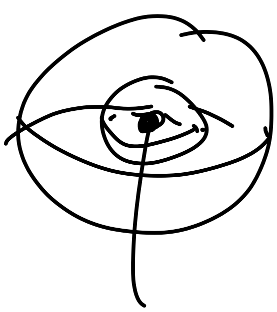
$$= \cos\varphi(\cos^2\varphi + \sin^2\varphi) = \cos\varphi$$

$$\iint_S \bar G\cdot\bar n\, dS = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} [\sin\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\, d\theta = \int_0^{2\pi} 2\, d\theta = 2\cdot 2\pi$$

$$= 4\pi$$

Merk: svaret er uavhengig av radius R.



summen av gravitasjonskreftene over et kuleskall som innholder jorda er konstant (uavhengig av radius)

Eks. Sylinder koordinater (polarkoordinater)

$z = f(x,y) = x^2 y$

$x^2 + y^2 \le 1$

A $\quad 0 \le \theta \le 2\pi$

$0 \le r \le 1$

$\bar r(x,y) = (x, y, f(x,y))$

Vektorfelt.

$\bar G(x,y,z) = (y, z, x^2 z)$

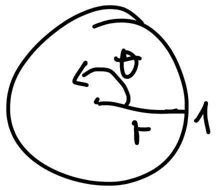
$$\frac{\partial \bar r}{\partial x}\times\frac{\partial \bar r}{\partial y} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

$$= (-2xy, -x^2, 1)$$

$$\bar G(\bar r(x,y)) = (y, x^2y, x^4y)$$

$$\iint_S \bar{G}\cdot\bar{n}\,dS = \iint_A (y, x^2y, x^4y)\cdot(-2xy, -x^2, 1)\,dx\,dy$$

$$= \iint_A -2xy^2 - \cancel{x^4y} + \cancel{x^4y}\,dx\,dy$$



$x = r\cos\theta$

$y = r\sin\theta$

$$= \iint_A -2xy^2\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 -2(r\cos\theta)(r\sin\theta)^2\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 -2r^4\cos\theta\sin^2\theta\,dr\,d\theta$$

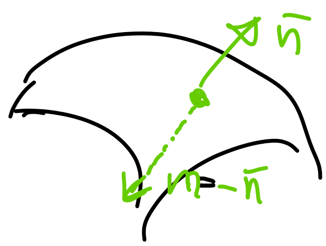
$$= -\frac{2}{5}\int_0^{2\pi}\sin^2\theta\cos\theta\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= -\frac{2}{5}\left[\frac{1}{3}\sin^3\theta\right]_0^{2\pi} = \underline{\underline{0}}$$

Kommentar om orienterbarhet.

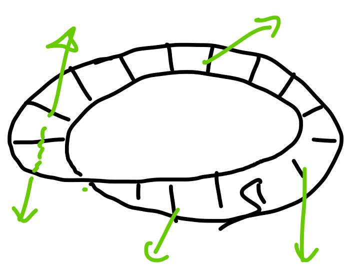


Innoverettet / Utoverettet

$\downarrow$ $\qquad$ $\downarrow$

Integral $= \div$ Integral

Problem:



Kun én side!

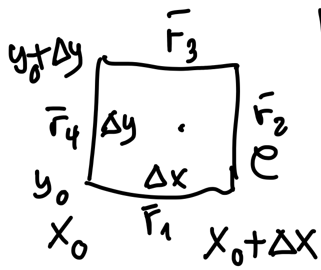
Ikke-orienterbar flate.

Möbiusbånd

Vrir en gang og limer sammen

Kjempeproblemer med $\iint_S \bar{G}\cdot\bar{n}\,dS$!!

Greens teorem (forhåndssnakk)

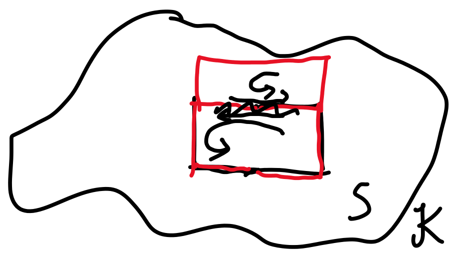


Vektorfelt $\bar{G} = (P, Q)$

$\oint_C \bar{G}\cdot d\bar{r} =$ skal regne ut.

$\approx \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\Delta x \Delta y$

la $\Delta x, \Delta y$ være veldig små



$$\boxed{\oint_K \bar{G}\cdot d\bar{r} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

Greens teorem

(generalisering av fundamentalteoremet for differensial og integralregning)