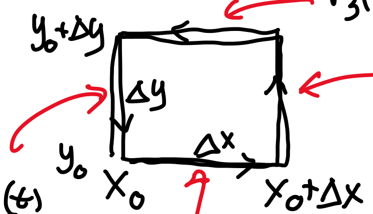
Forelesning, 10. mars 2021

onsdag 10. mars 2021 12:19

Greens teorem

$\bar{r}_3(t) = (t, y_0 + \Delta y)$



$\bar{r}_2(t) = (x_0 + \Delta x, t)$

$\bar{r}_4(t) = (x_0, t)$

$\bar{r}_1(t) = (t, y_0)$

Vektorfelt $\bar{G}(x,y) = (P, Q)$

Linjeintegralet av $\bar{G}$ rundt firkanten.

$$\bar{r}_1'(t) = \bar{r}_3'(t) = (1,0)$$

$$\bar{r}_2'(t) = \bar{r}_4'(t) = (0,1)$$

$$\boxed{(P,Q)(1,0) = P}$$

$$\int_{x_0}^{x_0+\Delta x} \bar{G}(t, y_0)\cdot(1,0)\,dt + \int_{y_0}^{y_0+\Delta y} \bar{G}(x_0+\Delta x, t)(0,1)\,dt$$

$$+ \int_{x_0+\Delta x}^{x_0} \bar{G}(t, y_0+\Delta y)\cdot(1,0)\,dt + \int_{y_0+\Delta y}^{y_0} \bar{G}(x_0, t)(0,1)\,dt$$

$$= \int_{x_0}^{x_0+\Delta x} P(t,y_0)\,dt + \int_{y_0}^{y_0+\Delta y} Q(x_0+\Delta x, t)\,dt$$

$$+ \int_{x_0+\Delta x}^{x_0} P(t, y_0+\Delta y)\,dt + \int_{y_0+\Delta y}^{y_0} Q(x_0, t)\,dt$$

$$= \int_{x_0}^{x_0+\Delta x} P(t,y_0) - P(t, y_0+\Delta y)\,dt + \int_{y_0}^{y_0+\Delta y} Q(x_0+\Delta x, t) - Q(x_0, t)\,dt$$

$$\boxed{\begin{array}{l}\text{Vi vet:} \\ \dfrac{\partial P}{\partial y}(t,y_0) \approx \dfrac{P(t, y_0+\Delta y) - P(t, y_0)}{\Delta y} \\ \text{Ved å la } \Delta y \to 0 \text{ får vi likhet}\end{array}}$$

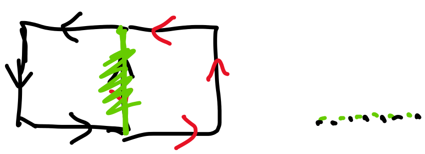
$$= \int_{x_0}^{x_0+\Delta x} -\underbrace{\frac{\partial P}{\partial y}(t,y_0)\,\Delta y}_{} \, dt + \int_{y_0}^{y_0+\Delta x} \frac{\partial Q}{\partial x}(x_0,t)\cdot \Delta x \, dt$$

betrakter som konstant i $[x_0, x_0+\Delta x]$

$$= -\frac{\partial P}{\partial y}(x_0,y_0)\,\Delta y \int_{x_0}^{x_0+\Delta x} dt + \frac{\partial Q}{\partial x}(x_0,y_0)\,\Delta x \int_{y_0}^{y_0+\Delta y} dt$$

$$= -\frac{\partial P}{\partial y}(x_0,y_0)\,\Delta y\,\Delta x + \frac{\partial Q}{\partial x}(x_0,y_0)\,\Delta x\,\Delta y$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)(x_0,y_0)\,\Delta x\,\Delta y$$



Fortsetter å dekke til med slike små firkanter til vi har dekket hele område

Omrisset til S: ∂S

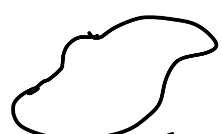
$$\boxed{\oint_{\partial S} \bar{G}\cdot d\bar{r} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

Greens teorem

To betingelser:

- Lukket kurve

  $\bar{r}(a) = \bar{r}(b)$

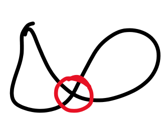


$\bar{r}(t)$

$a \leq t \leq b$

- Enkel kurve

IKKE

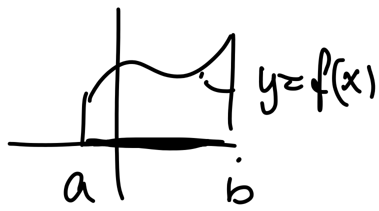


ingen samme punkter.

$a < s,t < b$ slik at

$$\bar{r}(s) = \bar{r}(t)$$
$$\Rightarrow s = t.$$

Enkel, lukket kurve i planet deler planet i to, innenfor og utenfor!



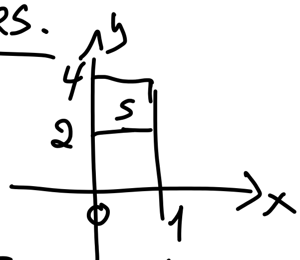
Anti-derivert $F(x)$

$$F'(x) = f(x)$$

$$\boxed{F(b) - F(a) = \int_a^b F'(x)\,dx}$$

$S = [a,b]$ $\quad \partial S = \{a,b\}$

Eks.



$$\bar{G}(x,y) = \left(\underset{P}{x + \tfrac{1}{3}y^3}, \underset{Q}{2x + y^2}\right)$$

$$\frac{\partial Q}{\partial x} = 2 \qquad \frac{\partial P}{\partial y} = y^2$$

$$\oint_{\partial S} \bar{G}\cdot d\bar{r} = \int_0^1 \int_2^4 2 - y^2 \,dy\,dx$$

$$= \int_0^1 \left[2y - \tfrac{1}{3}y^3\right]_2^4 dx$$

$$= \int_0^1 8 - \tfrac{1}{3}4^3 - 4 - \tfrac{1}{3}2^4\,dx$$

$$= -\frac{44}{3}$$

Anvendelser

$$\bar{G} = (P, Q)$$

Lager slik at $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$

F.eks $\boxed{Q = x \quad P = 0}$ $\leftarrow$ $\bar{G} = (0, x)$

eller $Q = 0 \quad P = -y$

eller noe annet.

$$\bar{r}(t) = (x(t), y(t))$$
$$a \le t \le b$$

$$\oint \bar{G}\cdot d\bar{r} = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dS$$

$$\oint_{\partial S} \vec{G} \cdot d\vec{r} = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dS$$

$$\int_a^b (0, x)(x'(t), y'(t))\,dt \qquad = \iint_S 1\,dS = \text{areal}(S)$$

$$\int_a^b x \cdot \underbrace{y'(t)}_{\frac{dy}{dt}}\,dt =: \int_{\partial S} x\,dy$$

$$\boxed{\int_{\partial S} x\,dy = \text{areal}(S)}$$

$$\boxed{P = -\tfrac{1}{2}y \quad Q = \tfrac{1}{2}x} \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$

Areal av ellipse S

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\bar{r}(t) = (a\cos t, b\sin t)$$

$$0 \leq t \leq 2\pi$$

$$\bar{r}'(t) = (-a\sin t, b\cos t)$$

$$\oint_{\partial S} \left(-\tfrac{1}{2}y, \tfrac{1}{2}x\right)\left(\frac{dx}{dt}, \frac{dy}{dt}\right)dt = \text{areal}(S)$$

$$\int_0^{2\pi} \left(-\tfrac{1}{2}b\sin t, \tfrac{1}{2}a\cos t\right) \cdot (-a\sin t, b\cos t)\,dt$$

$$= \int_0^{2\pi} +\tfrac{1}{2}ab \sin^2 t + \tfrac{1}{2}ab\cos^2 t \; dt$$

$$= \int_0^{2\pi} \tfrac{1}{2}ab\,dt = \tfrac{1}{2}ab \cdot 2\pi$$

$$= \underline{\underline{\pi ab}}$$

Tyngdepunkt

Vi vil regne ut tyngdepunktet.

Kan vi finne $\bar{G} = (P, Q)$ slik at $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x$?

$$\bar{x} = \frac{1}{\text{areal}} \iint_S x\,dS$$

$$\bar{y} = \frac{1}{\text{areal}} \iint_S y\,dS$$

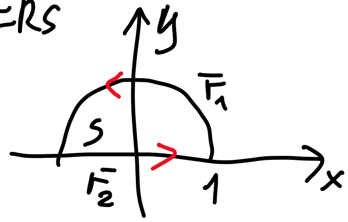
F.eks $P = -xy \quad Q = 0$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - (-x) = x$$

$P = 0 \quad Q = xy$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y - 0 = y$$

Eks



$$\text{areal}(S) = \frac{1}{2}\pi \cdot 1^2 = \underline{\underline{\frac{\pi}{2}}}$$

$$\oint_{\partial S} (0, xy) \cdot (x'(t), y'(t))\, dt = \iint_S y\, dS$$

$\bar{x} = 0$

$\bar{y} = ?$

$\bar{r}_1(t) = (\cos t, \sin t)$, $\bar{r}_1'(t) = (-\sin t, \cos t)$

$0 \le t \le \pi$

$\bar{r}_2(t) = (t, 0)$, $\bar{r}_2'(t) = (1, 0)$

$-1 \le t \le 1$

$$\iint_S y\, dS = \int_0^\pi (0, \cos t \cdot \sin t) \cdot (-\sin t, \cos t)\, dt$$

$$+ \int_{-1}^{1} \cancel{(0, t \cdot 0)} \cdot (1, 0)\, dt \quad = 0$$

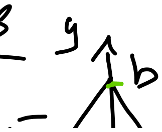
$$= \int_0^\pi \cos^2 t \cdot \sin t\, dt$$

$$= \left[-\frac{1}{3}\cos^3 t\right]_0^\pi = -\frac{1}{3}(\cos^3 \pi - \cos^3 0)$$

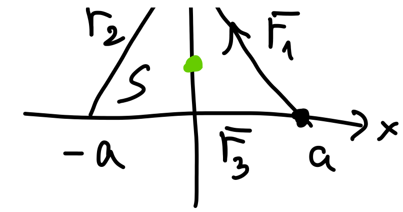
$$= -\frac{1}{3}(-1-1) = \frac{2}{3}$$

$$\bar{y} = \frac{1}{\text{areal}} \iint_S y\, dS = \frac{1}{\frac{1}{2}\pi} \cdot \frac{2}{3} = \underline{\underline{\frac{4}{3\pi}}}$$

Eks



Grunnlinje $2a$, høyde b



$$\text{areal} = \frac{1}{2}2a\cdot b = \underline{\underline{ab}}$$

$$\iint_S y\,dS = \oint_{\partial S} (0, xy)\,(x'(t), y'(t))\,dt$$

Tyngdepunkt: $\bar{x} = 0$

$\bar{y} = ?$

$\bar{r}_1(t) = \left(t,\ b - \frac{b}{a}t\right) \qquad \bar{r}_1'(t) = \left(1, -\frac{b}{a}\right)$

t går fra a til 0

$\bar{r}_2(s) = \left(s,\ b + \frac{b}{a}s\right) \qquad \bar{r}_2'(s) = \left(1, \frac{b}{a}\right)$

s går fra 0 til $-a$

$\bar{r}_3(x) = (x, 0) \quad -a \le x \le a \qquad \bar{r}_3'(x) = (1, 0)$

$$\iint_S y\,dS = \int_a^0 \left(0,\ t\left(b - \frac{b}{a}t\right)\right)\cdot\left(1, -\frac{b}{a}\right)dt$$

$$+ \int_0^{-a} \left(0,\ s\left(b + \frac{b}{a}s\right)\right)\cdot\left(1, \frac{b}{a}\right)ds$$

$$\cancel{+ \int_{-a}^{a} (0,0)\cdot(1,0)\,dx}$$

x: rettet etter forelesning

$$= \int_a^0 -t\left(b - \frac{b}{a}\right)\cancel{t}\,\frac{b}{a}\,dt + \int_0^{-a} s\left(b + \frac{b}{a}s\right)\frac{b}{a}\,ds$$

$s = -t$

$ds = -dt$

$s = 0 \Rightarrow t = 0$

$s = -a \Rightarrow t = a$

$$= -\frac{b}{a}\int_a^0 t^{\cancel{2}}\left(b - \frac{b}{a}t\right)dt$$

$$+ \int_0^a -t\left(b - \frac{b}{a}t\right)\frac{b}{a}(-dt)$$

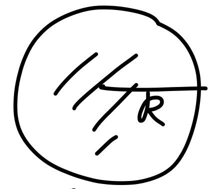
$$= \frac{b}{a}\int_0^a t^{\cancel{2}}\left(b - \frac{b}{a}t\right)dt + \frac{b}{a}\int_0^a t\left(b - \frac{b}{a}t\right)dt$$

$$= 2\frac{b}{a}\int_0^a bt^{\cancel{x}} - \frac{b}{a}t^{\overset{2}{\cancel{x}}}\,dt = \underline{\underline{\frac{1}{3}ab^2}}$$

$$\bar{y} = \frac{1}{\text{areal}}\iint_S y\,dS = \frac{1}{ab}\cdot\frac{1}{3}ab^2 = \underline{\underline{\frac{b}{3}}}$$

<u>Eks.</u>

$$\bar{G}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \qquad S:$$



$$= (P, Q)$$

$$\partial S: \bar{r}(t) = (R\cos t, R\sin t)$$
$$0 \le t \le 2\pi$$

$$D_{\bar{G}} = S \setminus \{(0,0)\}$$

$$\bar{G}(R\cos t, R\sin t) = \left(-\frac{\sin t}{R}, \frac{\cos t}{R}\right)$$

$$\oint_{\partial S} \bar{G}\cdot d\bar{r} = \int_0^{2\pi}\left(-\frac{\sin t}{R}, \frac{\cos t}{R}\right)\cdot(-R\sin t, R\cos t)\,dt$$

Radius = R

inneholder ikke R.

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = \int_0^{2\pi} dt = \underline{\underline{2\pi}}$$

$$P = \frac{-y}{x^2+y^2} \qquad \frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$Q = \frac{x}{x^2+y^2} \qquad \frac{\partial Q}{\partial x} = \qquad\qquad = \frac{y^2-x^2}{(x^2+y^2)^2}$$

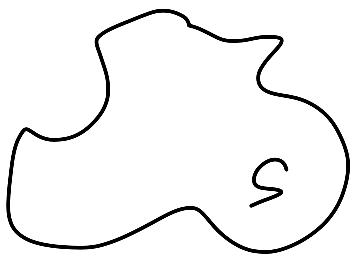
Konsekvens $\quad \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0 \quad$!!!

$$\iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dS = 0$$

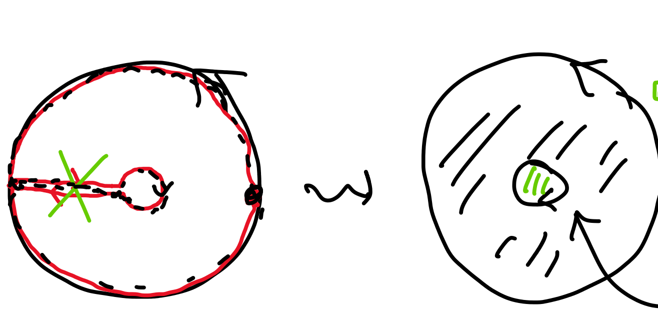
$$\underset{S}{\cdots}$$

$$\neq \oint_{\partial S} \bar{G} \cdot d\bar{r}$$

Er Greens teorem feil ?? Nei, fordi:



Forutsetning at $\bar{G}$ er vel-definert i hele S.

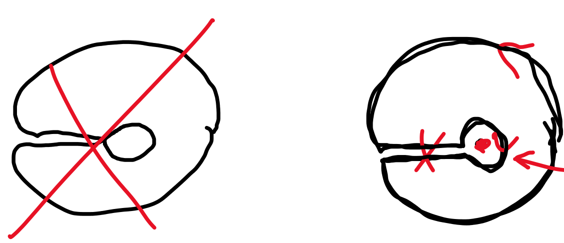


$$\oint_{\partial S_{ytre}} \bar{G} \cdot d\bar{r} = 2\pi$$

$$\oint_{\partial S_{indre}} \bar{G} \cdot d\bar{r} = -2\pi$$

$$\underset{\substack{\parallel \\ 2\pi - 2\pi}}{\oint_{\partial S} \bar{G} \cdot d\bar{r}} = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dS = 0$$

Samme vei indre og ytre



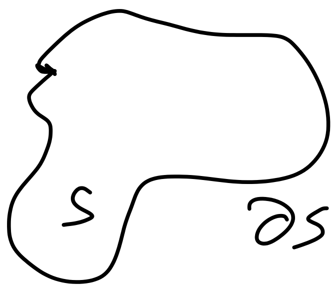
Indre radius kan gjøres veldig liten.

$$\bar{G} = \nabla \varphi = \left(\underset{\substack{\parallel \\ P}}{\frac{\partial \varphi}{\partial x}}, \underset{\substack{\parallel \\ Q}}{\frac{\partial \varphi}{\partial y}} \right) \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

Høyre-siden i Green = 0

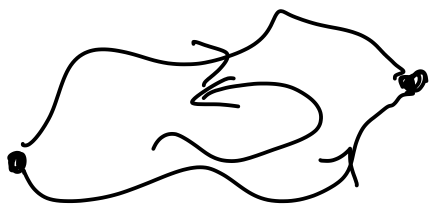
$\Rightarrow$ Venstre siden = 0

$\overline{G}$ konservativt



$\oint_{\partial S} \overline{G} \cdot dr = 0$

Linjeintegral er uavhengig av valgt vei



__Obs__: $\overline{G}$ må være definert i området mellom veien.