# Forelesning, 16. mars 2021

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## Div og curl.

Gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$f : \mathbb{R}^3 \to \mathbb{R}$ nabla

$\boxed{\nabla f}$ $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ Formell skrivemåte

Definisjoner:

$\bar{G} : \mathbb{R}^3 \to \mathbb{R}^3$

$\bar{G} = (P, Q, R)$

$$\operatorname{div}(\bar{G}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad : \mathbb{R}^3 \to \mathbb{R}$$

DIVERGENS

Eks $\bar{G} = (x^2 y, zy, x + zy)$

$$\operatorname{div}(\bar{G}) = \frac{\partial x^2 y}{\partial x} + \frac{\partial zy}{\partial y} + \frac{\partial (x+zy)}{\partial z}$$

$$= \underline{\underline{2xy + z + y}}$$

Formell skrivemåte

$$\operatorname{div}(\bar{G}) = \nabla \cdot \bar{G} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (P, Q, R)$$

$\boxed{\nabla \cdot \bar{G}}$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Curl / virvling : $\bar{G} : \mathbb{R}^3 \to \mathbb{R}^3$ $\bar{G} = (P, Q, R)$

Definisjon $\operatorname{curl}(\bar{G}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \begin{array}{l} \leftarrow \\ \leftarrow \nabla \\ \leftarrow \bar{G} \end{array}$$

Kryssprodukt: $\underline{v} = (v_1, v_2, v_3)$ $w = (w_1, w_2, w_3)$

$$\underline{v} \times \underline{w} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \begin{matrix} \leftarrow \\ \leftarrow \ \underline{v} \\ \leftarrow \ \underline{w} \end{matrix}$$

In analogi: $\operatorname{curl}(\bar{G}) = \nabla \times \bar{G}$

$$\boxed{\nabla \times \bar{G}}$$

**Eks.** $\bar{G} = (x^2 y, zy, x+2y)$

$$\operatorname{curl}(\bar{G}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & zy & x+2y \end{vmatrix} = (2-y, 0-1, 0-x^2)$$

$$= \underline{\underline{(2-y, -1, -x^2)}}$$

Bemerkning:

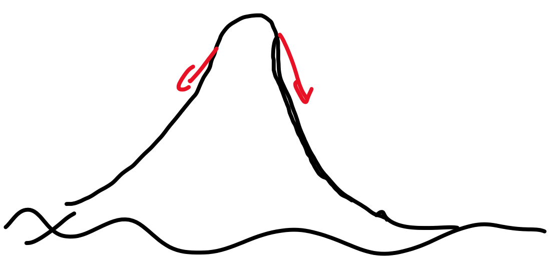
$$\nabla \cdot (\nabla f) = \nabla^2 f$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

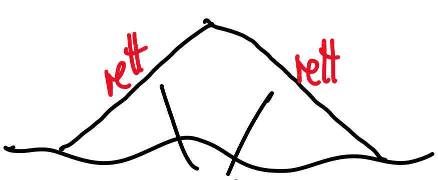
$$\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplace-operator

(Helt sentral i veldig mange diff. likninger.)



stabilisere seg




$\nabla^2 = 0$

To fine formler:

1) $\operatorname{curl}(\nabla f) = \nabla \times (\nabla f)$
$= (\nabla \times \nabla) f = 0$

2) $\operatorname{div}(\operatorname{curl}(\bar{G})) = \nabla \cdot (\nabla \times \bar{G})$

$$= 0$$

**Eks** $f(x,y,z) = x^2y + z$

$$\nabla f = (2xy,\ x^2,\ 1)$$

$$\operatorname{curl}(\nabla f) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 & 1 \end{vmatrix} = \bar{i}\frac{\partial 1}{\partial y} + \bar{j}\frac{\partial}{\partial z}(2xy) + \bar{k}\frac{\partial}{\partial x}(x^2)$$

$$-\bar{i}\frac{\partial}{\partial z}x^2 - \bar{j}\frac{\partial}{\partial x}1 - \bar{k}\frac{\partial}{\partial y}(2xy)$$

$$= 0 + 0 + 2x\cdot\bar{k} - 0 - 0 - 2x\bar{k}$$

$$= 0.$$

**Eks**

$$\bar{G} = (-xy^2,\ 0,\ -2yz - x^2)$$

$$\operatorname{curl}(\bar{G}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xy^2 & 0 & -2yz - x^2 \end{vmatrix}$$

$$= \bar{i}\frac{\partial}{\partial y}(-2yz - x^2) + \bar{j}\frac{\partial}{\partial z}(-xy^2) + \bar{k}\frac{\partial}{\partial x}0$$

$$-\bar{i}\frac{\partial}{\partial z}0 - \bar{j}\frac{\partial}{\partial x}(-2yz - x^2) - \bar{k}\frac{\partial}{\partial y}(-xy^2)$$

$$= -2z\cdot\bar{i} + 0 + 0 - 0 + 2x\bar{j} + \bar{k}2xy$$

$$= (-2z,\ 2x,\ 2xy)$$

$$\operatorname{div}(\operatorname{curl}(\bar{G})) = \nabla\cdot(-2z,\ 2x,\ 2xy)$$

$$= 0 + 0 + 0 = \underline{\underline{0}}$$

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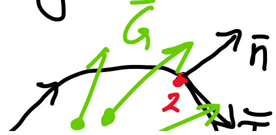
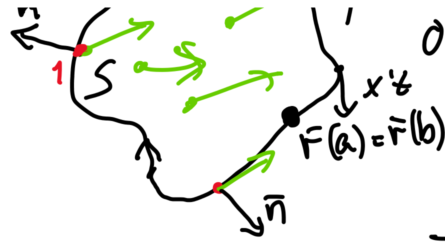
# Divergensteoremet

Hva sier dette




$\partial S: \bar{r}(t) = (x(t), y(t))$

i $\mathbb{R}^2$:




omriss $a \le t \le b$

$\bar T$: tangent

$\bar n$: normalvektor

$\bar G = (P, Q)$

$$\oint_{\partial S} \bar G \cdot \bar T \, ds = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Hva med $\oint_{\partial S} \bar G \cdot \bar n \, ds = ?$

Fluxen av $\bar G$ gjennom $\partial S$.
= Utstrømmingen av $\bar G$ gjennom $\partial S$

$\bar r(t) = (x(t), y(t))$

$\bar v(t) = \bar r'(t) = (x'(t), y'(t))$ Tangent

$(\bar r'(t) \perp)$ $\bar n(t) = (-y'(t), x'(t)) \frac{1}{v(t)}$ hvor $v(t) = \|\bar v(t)\|$

$\bar G = (P, Q)$

$ds = \frac{ds}{dt} \cdot dt = v(t) \cdot dt$

$$\oint_{\partial S} \bar G \cdot \bar n \, ds = \int_a^b \left(P(\bar r(t)), Q(\bar r(t))\right) \cdot \frac{(-y'(t), x'(t))}{\cancel{v(t)}} \cdot \cancel{v(t)}\, dt$$

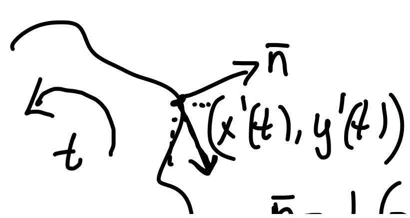
$$= \int_a^b -P(\bar r(t))\, \underset{\frac{dy}{dt}}{y'(t)} + Q(\bar r(t)) \cdot x'(t)\, dt$$

$$= \oint_{\partial S} -P\,dy + Q\,dx = \oint_{\partial S} Q\,dx - P\,dy$$

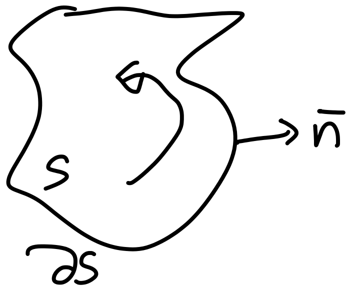
Greens teorem

$$= -\iint_S \frac{\partial(-P)}{\partial x} - \frac{\partial Q}{\partial y}\, dx\,dy = \iint_S \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dx\,dy$$

$$= \iint_S \operatorname{div}(\bar G)\, dx\,dy$$



$\bar{n} = \frac{1}{v(t)}(-y'(t), x'(t))$




$$\oint_{\partial S} \bar{G}\cdot\bar{n}\, ds = \iint_S \operatorname{div}(\bar{G})\, dx\,dy$$

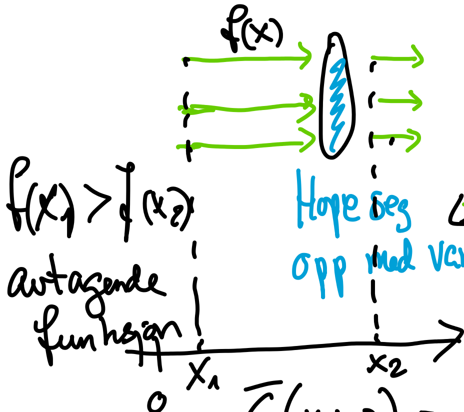
Fluxen av $\bar{G}$ gjennom $\partial S$

Integralet av divergensen over $S$.


Vann inn − vann ut

Positiv: Vannet hoper seg opp i $S$

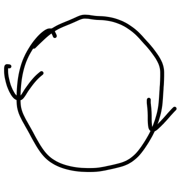
$\operatorname{div}(\bar{G})$: opphopning av feltet.




$(\operatorname{div}(\bar{G}) > 0)$

$\operatorname{div}(\bar{G}) < 0$

$$\bar{G}(x,y,z) = f(x)\cdot\bar{i} = (f(x),0,0)$$
$$\operatorname{div}(\bar{G}) = f'(x) < 0$$



# Divergensteoremet


$$\iint_{\partial V} \bar{G}\cdot\bar{n}\, dS = \iiint_V \nabla\cdot\bar{G}\, dV$$

Fluxen av $\bar{G}$ gjennom $\partial V$

Summen av divergensen i alle punkter inne i $V$.

Legeme: $V$

Overflate: $\partial V$

Vektor felt $\bar{G} = (P, Q, R)$

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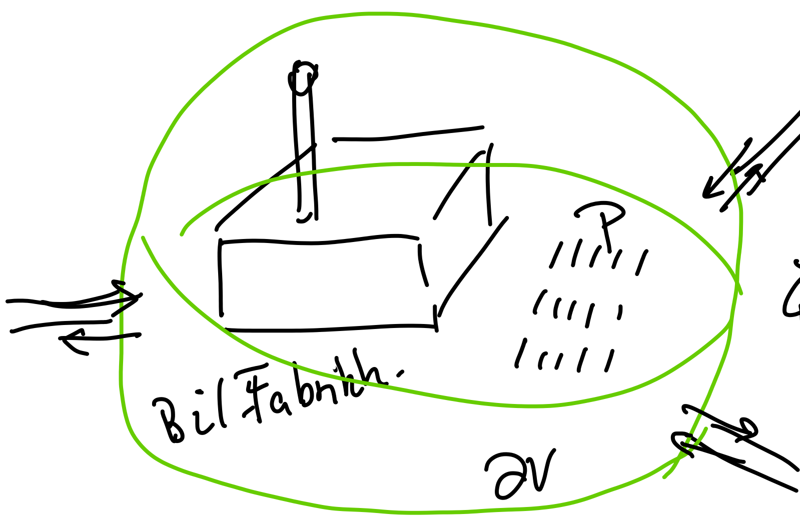


Vektorfeltet:

$\bar{G}$ : bilstrømmen

$\nabla \cdot \bar{G}$ : produksjon av biler

Flux av bilstrømmen
= Biler ut − biler inn

$$\iint_{\partial V} \bar{G} \cdot \bar{n} \, dS = \iiint_V \nabla \cdot \bar{G} \, dV$$

Totalproduksjon av biler inne i $V$.

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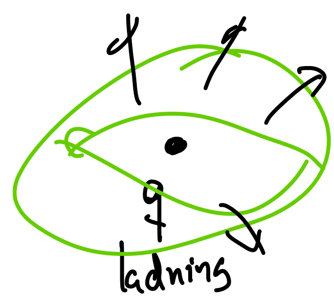


Ladningen ~ fluxen av feltet.

(divergensteoremet)