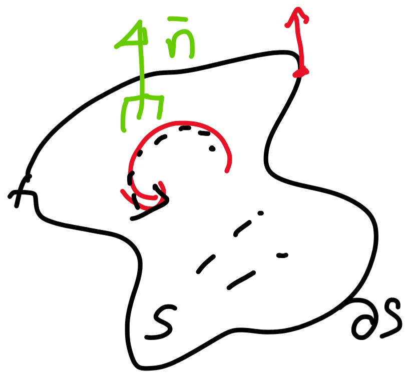
Forelesning, 17. mars 2021

onsdag 17. mars 2021 12:29

Curl / Stokes teorem



$$\int_{\partial S} \bar{G}\cdot d\bar{r} = \iint_S \operatorname{curl}(\bar{G})\cdot\bar{n}\, dS$$

$$\operatorname{curl}(\bar{G}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Flate i $\mathbb{R}^3$

$\bar{G} = (P, Q, R)$ vektorfelt

Generaliserer Greens teorem:

$S \subset \mathbb{R}^2$ ligger i planet.

$\bar{G} = (P, Q, 0)$

$\bar{n} = \bar{k}$

Venstre side svarer til venstre side i Greens teorem

Høyre side:

$$\operatorname{curl}(\bar{G}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \frac{\partial Q}{\partial z}\bar{i} + \frac{\partial P}{\partial z}\bar{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\bar{k}$$

$$\operatorname{curl}(\bar{G})\cdot\bar{n} = \operatorname{curl}(\bar{G})\cdot\bar{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$\bar{i}\cdot\bar{k} = \bar{j}\cdot\bar{k} = 0$

$\bar{n}\cdot\bar{k} = 1$

$$\boxed{\oint_{\partial S} G\cdot d\bar{r} = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy}$$

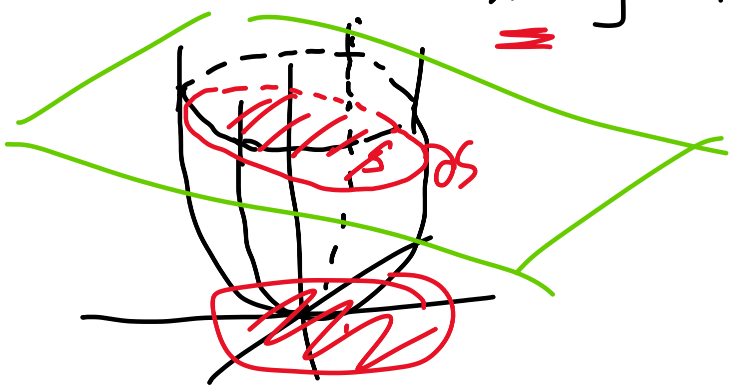
Greens teorem

Stokes teorem i planet = Greens teorem

Eks

$$\bar{G}(x,y,z) = (zy + e^{x^2},\ xyz,\ xy + \cos^3 \cancel{x}\, z)$$

Flaten: Paraboloide $z = x^2 + y^2$
Plan: $z = 2x - 4y + 4$



Vanskelighet:
Beskrive S og ∂S

Regne ut $\int_{\partial S} \bar{G}\cdot d\bar{r}$ ved Stokes teorem

Trenger:
$$\operatorname{curl}(\bar{G}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zy + e^{x^2} & xyz & xy + \cos^3 \cancel{x}\, z \end{vmatrix}$$

$$= x\bar{i} + y\bar{j} + yz\bar{k} - xy\bar{i} - y\bar{j} - z\cdot\bar{k}$$

$= (x - xy, y - y, yz - z)$

$= (x - xy, 0, yz - z) = (x - xy, 0, z(y-1))$

Setter $z = x^2 + y^2$ lik med $z = 2x - 4y + 4$

$$x^2 + y^2 = 2x - 4y + 4$$

(Litt regning) samme som

$$(x-1)^2 + (y+2)^2 = 3^2 \quad \text{(trykkfeil i boka)}$$

sirkel med sentrum i $(1,-2)$ og radius 3

Parametrisering av planet:

$$\bar{r}(x,y) = (x, y, \underline{2x - 4y + 4})$$

Flatenormal: $\frac{\partial \bar{r}}{\partial x} = (1, 0, 2)$: $\frac{\partial \bar{r}}{\partial x} \times \frac{\partial \bar{r}}{\partial y}$

$\frac{\partial \bar{r}}{\partial y} = (0, 1, -4)$

$$\bar{n} = \frac{1}{\text{lengden}} \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & -4 \end{vmatrix} = (-2, 4, 1) \frac{1}{\sqrt{21}}$$

$$\text{curl}(\bar{G})(\bar{r}(x,y)) = (x - xy, 0, (y-1)(2x - 4y + 4)).$$

$$\iint_S \text{curl}(\bar{G}) \cdot \bar{n} \, dS = \iint (x - xy, 0, (y-1)(2x - 4y + 4)) (-2, 4, 1) \cancel{\frac{1}{\sqrt{21}}} \cancel{\sqrt{21}} \, dx \, dy$$

$$= \iint_S -2x + 2xy + 2xy - 4y^2 + 4y - 2x + 4y - 4 \, dx \, dy$$

$$= \iint_S \left(-4x + 8y + 4xy - 4y^2 - 4\right) dx\,dy$$

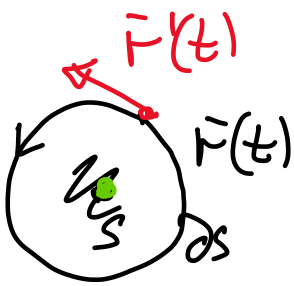
over $(x-1)^2 + (y+2)^2 \le 3^2$

Parametriser:

$x = 1 + r\cos\theta \qquad 0 \le r \le 3$

$y = -2 + r\sin\theta \qquad 0 \le \theta \le 2\pi$

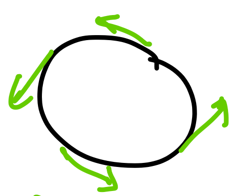
Svar $= \underline{\underline{-513\pi}}$



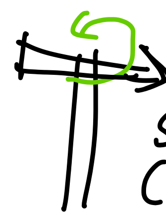
$$\int_{\partial S} \bar G \cdot d\bar r$$

$\bar G$ vektorfelt

Eks 1

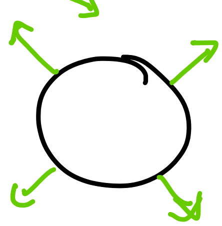


$$\int_{\partial S} \bar G \cdot d\bar r > 0$$



sirkulere (virle)

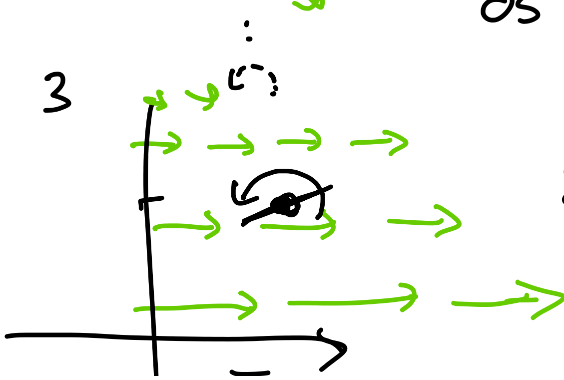
2



$$\int_{\partial S} \bar G \cdot d\bar r = 0$$

stå stille

3



Stanga vil sirkuler

$$\mathrm{curl}(\bar G) = \begin{vmatrix} \bar i & \bar j & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

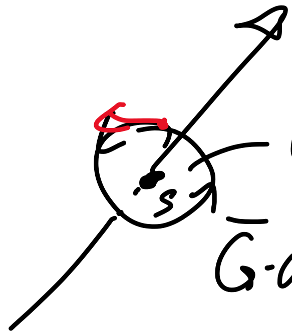
$\bar G = \left(\frac{1}{y}, 0, 0\right)$

$\mathrm{curl}(\bar G) = \begin{vmatrix} \cdots \\ \frac{1}{y} & 0 & 0 \end{vmatrix}$

$= \left(0, 0, +\frac{1}{y^2}\right)$ $\bar k$-retning

$\frac{1}{y^2} > 0$

virvlar oppover



$\bar G$ = konstant på S

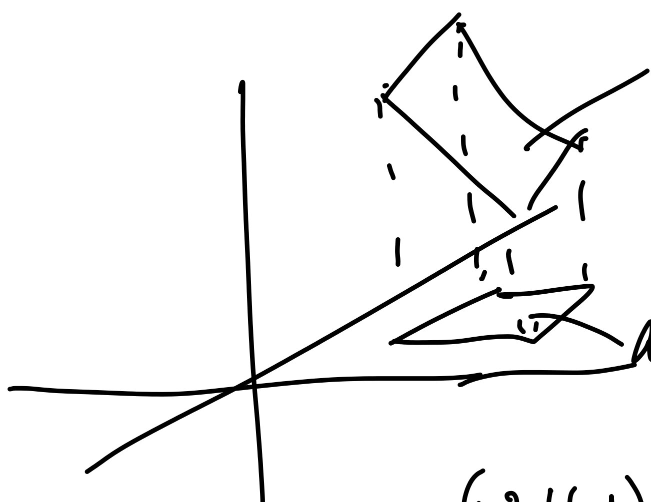
$\bar G \cdot d\bar r \sim \mathrm{curl}(\bar G)$

I hvilken grad $\bar G$ og $\bar T$ går i samme retning

$$\int_{\partial S} \bar G \cdot d\bar r = 2\pi R \cdot (\bar G \cdot \bar T)$$

$$= \iint_S \mathrm{curl}(\bar G) \cdot \bar n \, dS = \pi R^2 \cdot \|\mathrm{curl}(\bar G)\|$$

= størrelsen på virvlingen



$$\frac{(-2,4,1)}{\cancel{\sqrt{21}}} \cdot \cancel{\sqrt{21}}$$

enhets-normalvektor

forstørrelses-faktor (Jacobi)