Lineæravbildninger Affinavbildninger
(- funksjoner)

$$F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{pmatrix}$$

n-søyle m-søyle

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-y+3z \\ x-4y+2z \end{pmatrix} \qquad F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ x^2y+z \\ y\sin(x^2z) \\ 1 \end{pmatrix}$$

$T: \mathbb{R}^3 \to \mathbb{R}^2$ $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

Lineær Ikke lineær

Formell definisjon

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$: $\forall a \in \mathbb{R}, \ \forall \underline{x}, \underline{y} \in \mathbb{R}^n$

lineær avbildning

1) $T(\underline{x}+\underline{y}) = T(\underline{x}) + T(\underline{y})$

2) $T(a \cdot \underline{x}) = aT(\underline{x})$

(Alternativ: $T(a\underline{x}+b\underline{y}) = aT(\underline{x}) + bT(\underline{y})$)

Viktig resultat

a) $\underset{m\times n\text{-matrise}}{A} \Rightarrow T: \mathbb{R}^n \to \mathbb{R}^m$ gitt ved $T(\underline{x}) = A\underline{x}$ er en lineæravbildning

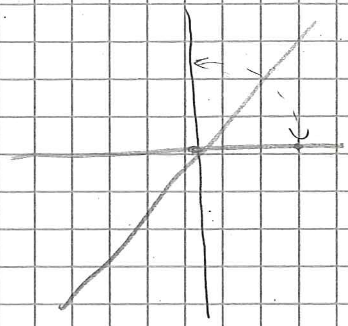
b) $T: \mathbb{R}^n \to \mathbb{R}^m$ lineær avb. $\Rightarrow \exists \underset{m\times n\text{-matrise}}{A}$ slik at $T(\underline{x}) = A\underline{x}$ $\forall \underline{x} \in \mathbb{R}^n$

Bevis: se læreboka

Eksempel (Speiling)

$$S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

speiling om linja $y=x$



$$S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \quad S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

betyr at $S\begin{pmatrix}a\\b\end{pmatrix} = a\,S\begin{pmatrix}1\\0\end{pmatrix} + b\,S\begin{pmatrix}0\\1\end{pmatrix}$

$$= a\begin{pmatrix}0\\1\end{pmatrix} + b\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}b\\a\end{pmatrix}$$

$$S(\underline{x}) = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\underline{x}$$

Eksempel (Rotasjon)

$$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

Rotasjon med vinkel θ

(pos omløpsretning)

$$R_\theta\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \quad R_\theta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

Betyr at $R_\theta(\underline{x}) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\underline{x}$.

Merk: Vinkelmåler additiv

$$R_{\theta_1+\theta_2} = R_{\theta_1}\cdot R_{\theta_2}$$

$$\begin{pmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{pmatrix} = \begin{pmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix}\begin{pmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{pmatrix}$$

$$\Longrightarrow \cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

Addisjonslovene for cos/sin.

Linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ (samme eksponent)

Fikspunkt $T(\underline{x}) = \underline{x}$

F.eks – rotasjonssenter
– speilingsakse, eller -plan

En litt friere definisjon.

Egenverdi
-vektorer $T(\underline{x}) = \lambda \underline{x} \qquad \lambda \in \mathbb{R}$

$T(\underline{x}) = A\underline{x} \rightsquigarrow A\underline{x} = \lambda \underline{x}$

Eksempel

$$A = \begin{pmatrix} 1 & -8 \\ -2 & 1 \end{pmatrix} \qquad A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \end{pmatrix} = \underbrace{5}_{\text{Egenverdi}} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \end{pmatrix} = -3\underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\text{egenvektor}}$$

$$T\left(a\begin{pmatrix} 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = a\,T\begin{pmatrix} 2 \\ -1 \end{pmatrix} + b\,T\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= a \cdot 5\begin{pmatrix} 2 \\ -1 \end{pmatrix} + b(-3)\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Kommer senere: Alle vektorer i $\mathbb{R}^2$ kan skrives på formen $a\begin{pmatrix} 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $a, b \in \mathbb{R}$

Hva med rotasjon? Ingen vektor kan oppfylle $R_\theta(\underline{x}) = \lambda \underline{x}$

Men.
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos\theta - i\sin\theta \\ \sin\theta + i\cos\theta \end{pmatrix}$$

$$= (\cos\theta - i\sin\theta)\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Affin avbildning

$F: \mathbb{R}^n \to \mathbb{R}^m$: $\exists A$, $\underline{c} \in \mathbb{R}^m$ slik at
affin $\qquad$ m×n-matrise

$$F(\underline{x}) = A\underline{x} + \underline{c}$$

$\qquad$ matrise $\qquad$ konstantledd

For $n = m$

Affin avbildning = Lineæravbildning + Translasjon

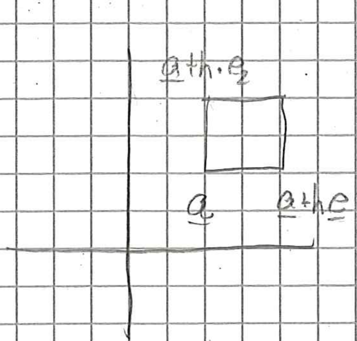
$$A\underline{x} \qquad + \underline{c}$$

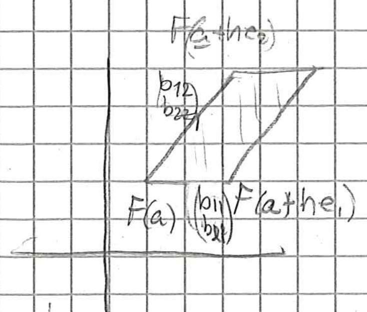
Rette linjer $\longrightarrow$ Rette linjer
(Men ikke gjennom origo)

Eks

Linje: $\mathcal{L}$: $\underline{r}(t) = \underline{a} + t\,\underline{b}$

$$\begin{aligned} F(\underline{r}(t)) &= A\,\underline{r}(t) + \underline{c} \\ &= A\cdot\underline{a} + tA\underline{b} + \underline{c} \\ &= \underbrace{(A\cdot\underline{a} + \underline{c})}_{= F(\underline{a})} + t\cdot(A\underline{b}) \end{aligned}$$





areal $\left|\det\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right|$

$$F\underline{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} x + \underline{c}$$

$\Rightarrow$ arealet endrer seg med faktor $|\det(A)|$.