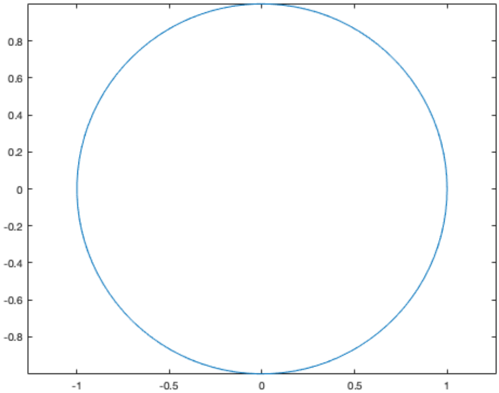
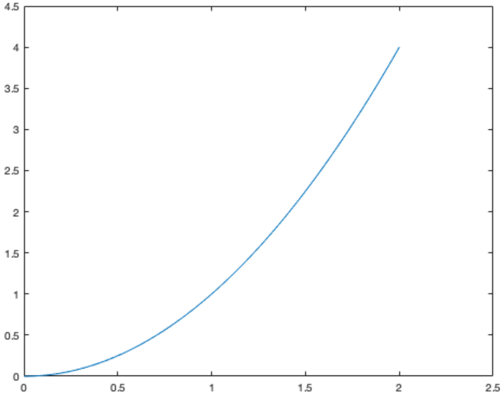
MAT 1110, 24. januar 2022

$*$ Parametriserte kurver

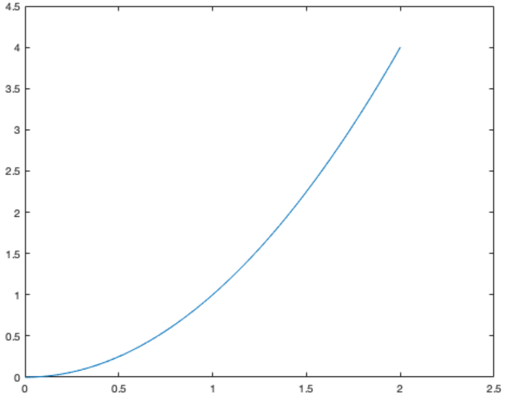
Arne B. Sletsjøe
Universitetet i Oslo



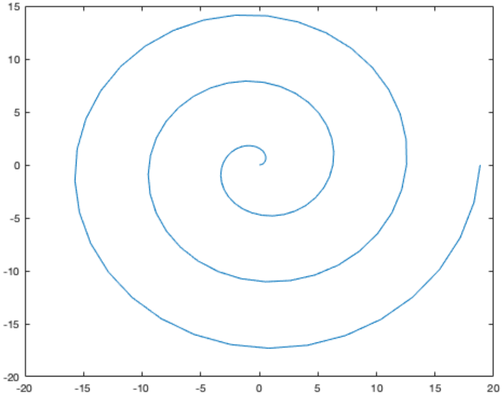
$$x(t) = \cos t,\ y(t) = \sin t \text{ hvor } 0 \leq t \leq 2\pi$$



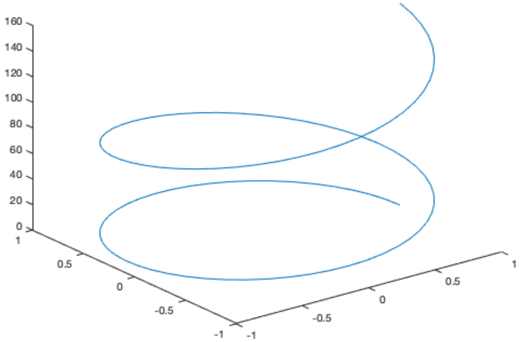
$x(t) = t,\ y(t) = t^2$ hvor $0 \leq t \leq 2$



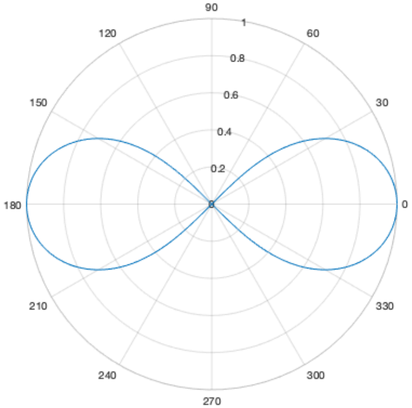
$$x(t) = t^2,\ y(t) = t^4 \text{ hvor } 0 \leq t \leq \sqrt{2}$$



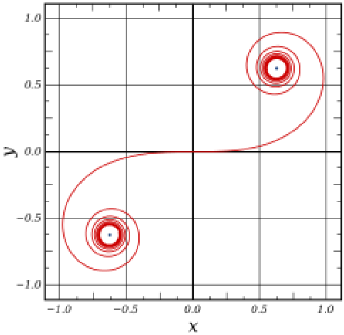
$$x(t) = t\cos t,\ y(t) = t\sin t \text{ hvor } 0 \le t \le 6\pi$$



$$x(t) = \cos t,\ y(t) = \sin t,\ z(t) = t^2 \text{ hvor } 0 \leq t \leq 4\pi$$



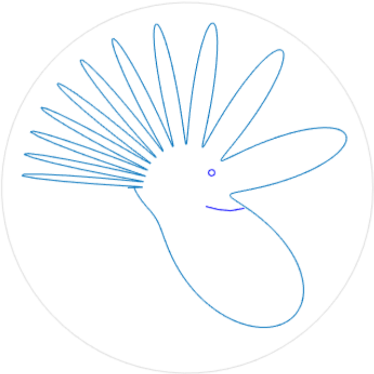
$$\rho(\theta) = \sqrt{\cos 2\theta}, \text{ hvor } 0 \leq \theta \leq 2\pi$$



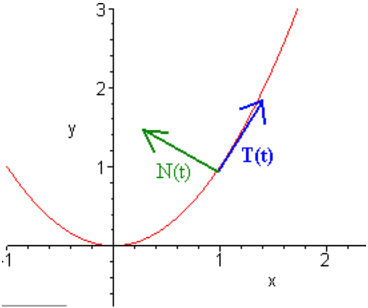
$$x(t) = a\sqrt{\pi} \int_0^t \cos \frac{\pi \xi^2}{2} \, d\xi$$

$$y(t) = a\sqrt{\pi} \int_0^t \sin \frac{\pi \xi^2}{2} \, d\xi$$

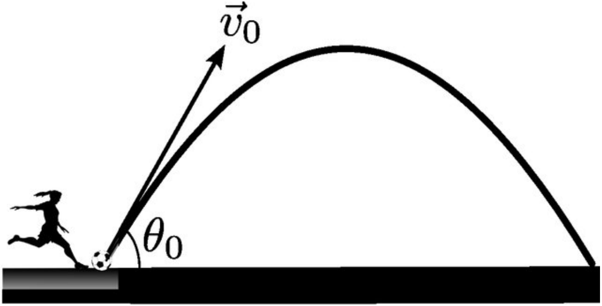
$t = \frac{\ell}{a\sqrt{\pi}}$ hvor ℓ er buelengden til klotoiden regnet fra origo.

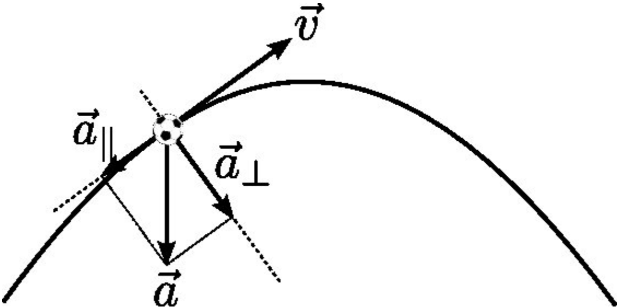


$$\rho(\theta) = \sin 2^{\theta} - 1.7 \text{ hvor } 0 \leq \theta \leq 2\pi$$



3
2
1
y
-1
1
2
x
N(t)
T(t)


$\vec{v}_0$
θ_0



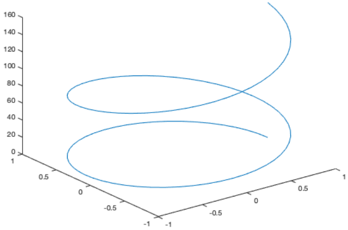
$\vec{v}$
$\vec{a}_\parallel$
$\vec{a}_\perp$
$\vec{a}$

Definisjon

En **parametrisert kurve** *i* $\mathbb{R}^n$ *er en kontinuerlig funksjon* $\mathbf{r} : I \to \mathbb{R}^n$ *hvor* $I \subset \mathbb{R}$ *er et intervall. Vi skriver ofte funksjonen på komponentform;*

$$\mathbf{r}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

Eksempel



$$\mathbf{r}(t) = (\cos t, \sin t, t^2),\ 0 \leq t \leq 4\pi$$

Definisjon

Anta at funksjonene $x_1, \dots, x_n$ *er deriverbare i punktet* t. *Da sier vi at den parametriserte kurven* $\mathbf{r}(t) = (x_1(t), x_2(t), \dots, x_n(t))$ *er* **deriverbar** *i* t *og at den* **deriverte** *er*

$$\mathbf{v}(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))$$

Hvis $\mathbf{r}(t)$ *representerer posisjonen til en gjenstand ved tiden* t, *sier vi at* $\mathbf{v}(t)$ *representerer* **hastigheten** *til gjenstanden.*

Absoluttverdien av hastighet kaller vi **fart**; $v(t) = |\mathbf{v}(t)|$.

Eksempel

Betrakt kurven

$$\mathbf{r}(t) = (t\cos t, t\sin t), \qquad 0 \leq t \leq 6\pi$$

Vi har

$$\mathbf{v}(t) = (\cos t - t\sin t, \sin t + t\cos t)$$

og $v(t) = \sqrt{1+t^2}$.

Definisjon

Anta a funksjonene $x_1, x_2, \ldots, x_n$ *er deriverbare med kontinuerlige deriverte. Da er* **buelengden** *fra* $\mathbf{r}(a)$ *til* $\mathbf{r}(b)$ *av den parametriserte kurven* $\mathbf{r}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *gitt ved*

$$\begin{aligned} \mathbf{L}(a, b) &= \int_a^b \sqrt{x_1'(t)^2 + x_2'(t)^2 + \cdots + x_n'(t)^2}\, dt \\ &= \int_a^b v(t)\, dt \end{aligned}$$

Eksempel

Vi betrakter klotoiden

$$x(t) = a\sqrt{\pi}\int_0^t \cos\frac{\pi\xi^2}{2}\,d\xi \qquad y(t) = a\sqrt{\pi}\int_0^t \sin\frac{\pi\xi^2}{2}\,d\xi$$

Det gir

$$x'(t) = a\sqrt{\pi}\cos\frac{\pi t^2}{2} \qquad y'(t) = a\sqrt{\pi}\sin\frac{\pi t^2}{2}$$

hvor $t = \frac{\ell}{a\sqrt{\pi}}$ *og* ℓ *er buelengden til klotoiden regnet fra origo. Dette gir konstant fart*

$$v(t) = |\mathbf{v}(t)| = \sqrt{(x'(t))^2 + (y'(t))^2} = a\sqrt{\pi}$$

Setning

La $\mathbf{r}(t)$, $\mathbf{r}_1(t)$ *og* $\mathbf{r}_2(t)$ *være deriverbare parametriserte kurver og* $u(t)$ *en deriverbar funksjon. Da har vi*

(i) $(\mathbf{r}_1(t) + \mathbf{r}_2(t))' = \mathbf{r}_1'(t) + \mathbf{r}_2'(t)$

(ii) $(k \cdot \mathbf{r}_1(t))' = k \cdot \mathbf{r}_1'(t)$, $k \in \mathbb{R}$

(iii) $(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t))' = \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t)$

(iv) $(\mathbf{r}_1(t) \times \mathbf{r}_2(t))' = \mathbf{r}_1'(t) \times \mathbf{r}_2(t) + \mathbf{r}_1(t) \times \mathbf{r}_2'(t)$ *(for kurver i* $\mathbb{R}^3$*)*

(v) $(u(t)\mathbf{r}(t))' = u'(t)\mathbf{r}(t) + u(t)\mathbf{r}'(t)$

Bevis.

Hvis vi betrakter uttrykkene komponentvis følger resultatet av tilsvarende resultat for funksjoner i en variabel. $\square$

Korollar

Dersom $|\mathbf{r}(t)|$ *er konstant, så er* $\mathbf{r}(t)$ *og* $\mathbf{r}'(t)$ *ortogonale.*

Bevis.

At $|\mathbf{r}(t)|$ er konstant betyr at $\mathbf{r}(t) \cdot \mathbf{r}(t) = C$. Vi bruker (iii) over og at den deriverte av en konstant er 0, og får

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

$\square$

Eksempel

Vi betrakter en kurve på en kuleflate med radius 1 gitt ved

$$\mathbf{r}(t) = (\cos(\theta(t))\cos(\phi(t)), \sin(\theta(t))\cos(\phi(t)), \sin(\phi(t)))$$

hvor $0 \leq \theta(t) \leq 2\pi$ *og* $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$. *Vi har* $|\mathbf{r}(t)| = 1$ *for alle* t *og* $\mathbf{r}(t)$ *og* $\mathbf{r}'(t)$ *er ortogonale. Det betyr at hastigheten* $\mathbf{v}(t)$ *er tangentiell til kuleflaten.*

Definisjon

La

$$\mathbf{a}(t) = \mathbf{v}'(t) = (x_1''(t), x_2''(t), \ldots, x_n''(t))$$

Hvis $\mathbf{r}(t)$ *representerer posisjonen til en gjenstand ved tiden* t*, sier vi at* $\mathbf{a}(t)$ *representerer* **akselerasjonen** *til gjenstanden.*

Vi kaller $a(t) = v'(t)$ *for* **baneakselerasjonen** *til gjenstanden.*

Anta at $\mathbf{v}(t) \neq 0$. Da definerer vi **enhetstangentvektoren** $\mathbf{T}(t)$ ved

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)}$$

Vi har $|\mathbf{T}(t)| = 1$ og $\mathbf{v}(t) = v(t)\mathbf{T}(t)$. Det betyr at

$$\mathbf{a}(t) = \mathbf{v}'(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) = a(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$$

Setning

Anta at $\mathbf{v}(t) \neq 0$. *Da kan vi dekomponere akselerasjonene* $\mathbf{a}(t)$ *i to ortogonale komponenter*

$$\mathbf{a}(t) = a(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$$

der $a(t)\mathbf{T}(t)$ *er parallell med tangenten og* $v(t)\mathbf{T}'(t)$ *står normalt på tangenten.*

Eksempel

Vi fortsetter å betrakte klotoiden

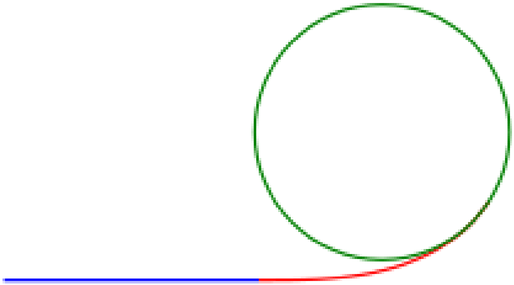
$$x(t) = a\sqrt{\pi}\int_0^t \cos\frac{\pi\xi^2}{2}\,d\xi \qquad y(t) = a\sqrt{\pi}\int_0^t \sin\frac{\pi\xi^2}{2}\,d\xi$$

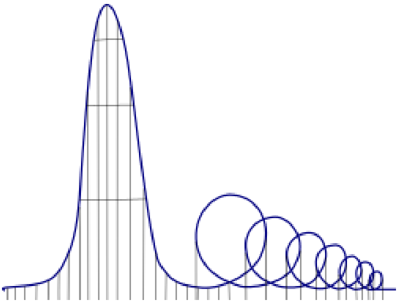
Vi har

$$x''(t) = -a\sqrt{\pi}\pi t\sin\frac{\pi t^2}{2} \qquad y''(t) = a\sqrt{\pi}\pi t\cos\frac{\pi t^2}{2}$$

Siden farten er konstant vil baneakselerasjonen være 0, og akselerasjonen står normalt på hastigheten, med absoluttverdi

$$a(t) = |\mathbf{a}(t)| = \sqrt{(x''(t))^2 + (y''(t))^2} = a(\pi)^{\frac{3}{2}}t = \pi\ell$$





The Euthanasia Coaster would kill its passengers through prolonged cerebral hypoxia, or insufficient supply of oxygen to the brain. The ride's seven inversions would inflict 10 g (g-force) on its passengers for 60 seconds – causing g-force related symptoms starting with gray out through tunnel vision to black out and eventually g-LOC (g-force induced loss of consciousness). Subsequent inversions or another run of the coaster would serve as insurance against unintentional survival of more robust passengers.

(Wikipedia)

Definisjon

La $f : \mathbb{R}^n \to \mathbb{R}$ *være en deriverbar funksjon. Da er* **gradienten** ∇f *til* f *gitt ved*

$$\nabla f(x_1, \ldots, x_n) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

Eksempel

Funksjonen

$$f(x_1, x_2, x_3) = x_1 x_2 + x_3$$

har gradient

$$\nabla f(x_1, x_2, x_3) = (x_2, x_1, 1)$$

Setning

Anta at den parametriserte kurven $\mathbf{r}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *er deriverbar i punktet* t*, og at skalarfeltet* $f : \mathbb{R}^n \to \mathbb{R}$ *er deriverbart i punktet* $\mathbf{r}(t)$*. Da er den sammensatte funksjonen* $u(t) = f(\mathbf{r}(t))$ *deriverbar i* t *og*

$$u'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

Eksempel

Vi betrakter sammensetningen $u(t) = f(\mathbf{r}(t))$ *av*

$$f(x_1, x_2, x_3) = x_1 x_2 + x_3$$

og

$$\mathbf{r}(t) = (\cos t, \sin t, t^2)$$

Den deriverte:

$$\begin{aligned}
u'(t) &= \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \\
&= (x_2, x_1, 1) \cdot (-\sin t, \cos t, 2t) \\
&= (\sin t, \cos t, 1) \cdot (-\sin t, \cos t, 2t) \\
&= -\sin t \sin t + \cos t \cos t + 2t \\
&= \cos(2t) + 2t
\end{aligned}$$

Setning

La $A \subset \mathbb{R}^n$ og anta at $f : A \to \mathbb{R}$ er deriverbar i et område som inneholder linjestykket mellom $\mathbf{a}$ *og* $\mathbf{b}$ *i* $\mathbb{R}^n$. *Da finnes et punkt* $\mathbf{c}$ *på linjestykket fra* $\mathbf{a}$ *til* $\mathbf{b}$ *slik at*

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$

Bevis.

La $g(t) = f(\mathbf{r}(t))$ og $\mathbf{r}(t) = t\mathbf{a} + (1-t)\mathbf{b}$ hvor $t \in [0,1]$. Da vil

$$g'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \nabla f(\mathbf{r}(t)) \cdot (\mathbf{b} - \mathbf{a})$$

og setningen følger fra den vanlige middelverdisetningen for funksjoner i en variabel. $\square$