MAT 1110, 7. februar 2022

∗ Skalar- og vektorfelt,
grafisk framstilling

∗ Strømningskurver

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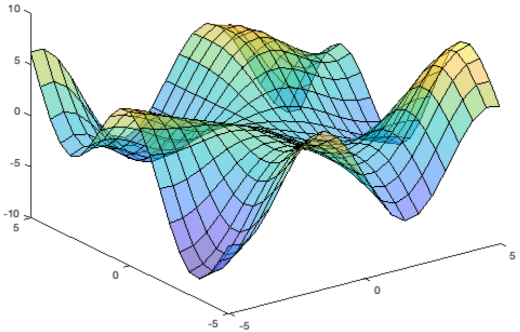
Skalarfelt

Definisjon

(i) *En funksjon* $f : \mathbb{R}^n \to \mathbb{R}$ *kalles et* **skalarfelt**.

(ii) *En funksjon* $f : \mathbb{R}^n \to \mathbb{R}^n$ *(hvor* $n > 1$*) kalles et* **vektorfelt**.



$$f(x, y) = y \sin x - x \cos y,$$

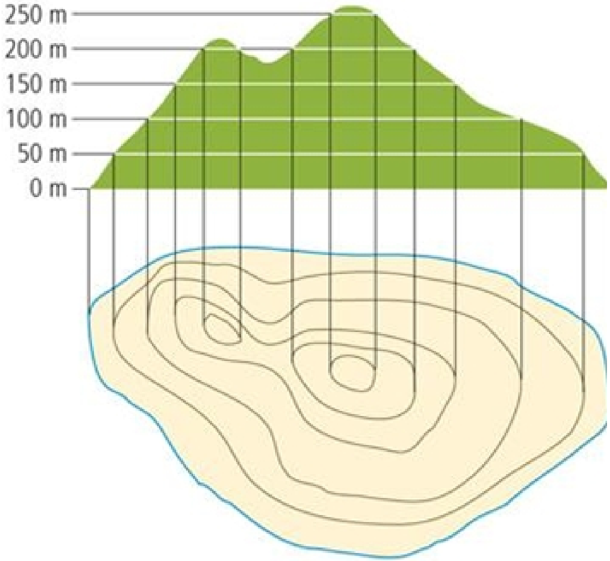
Definisjon

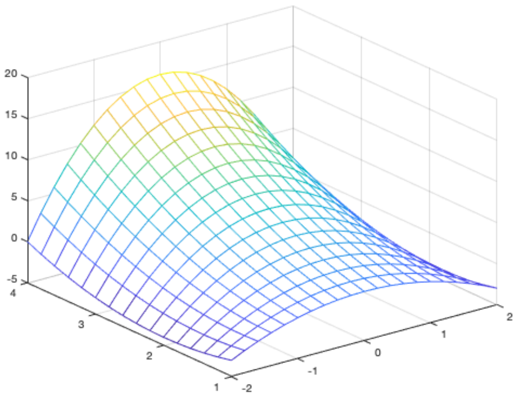
La $f : \mathbb{R}^n \to \mathbb{R}$ *være et skalarfelt. Mengden*

$$N_c = \{\mathbf{x} \in \mathbb{R}^n \,|\, f(\mathbf{x} = c\}$$

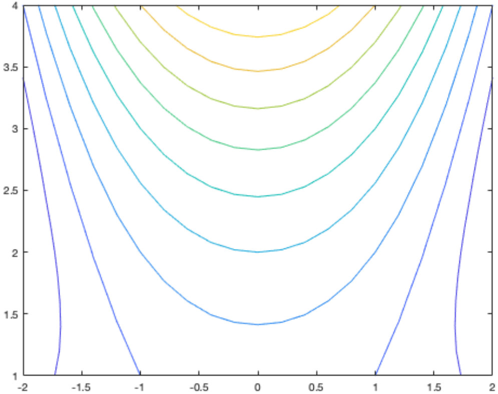
kalles en **nivåmengde** *for* f*. Nivåmengden har dimensjon* $n - 1$*.*

I planet kalles nivåmengdene for **nivåkurver***, mens de i rommet kalles* **nivåflater***.*

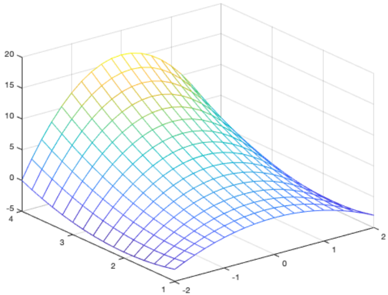

250 m
200 m
150 m
100 m
50 m
0 m

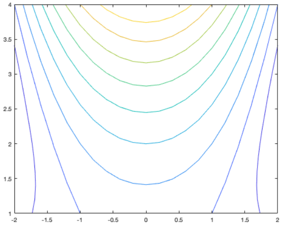


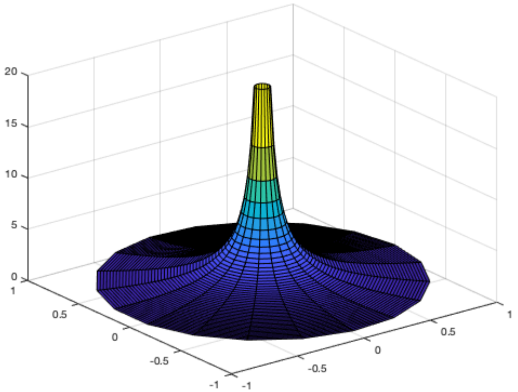
$$z = f(x, y) = y^2 - x^2 y$$



$$z = f(x, y) = y^2 - x^2 y = C$$


20
15
10
5
0
-5
4
3
2
1
-2
-1
0
1
2


4
3.5
3
2.5
2
1.5
1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2



Polarkoordinater:

$$x = r\cos\theta, \quad y = r\sin\theta$$

hvor $r > 0$ og $0 \leq \theta \leq 2\pi$.

Dette er grafen til $f(r,\theta) = \frac{1}{r}$.

Koordinater i $\mathbb{R}^3$:

(i) Kartesiske koordinater (rettvinklede koordinater): (x, y, z)

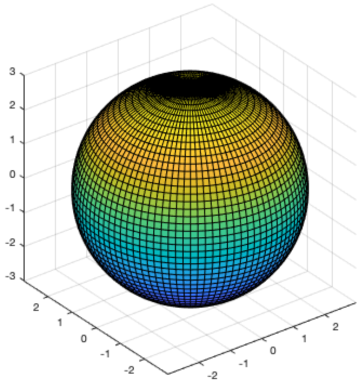
(ii) Sylinderkoordinater: (r, θ, z) (Polarkoordinater/kartesiske koordinater)

(iii) Sfæriske koordinater (eller kulekoordinater): (r, θ, ϕ) (Polar/polar-koordinater)

Kuleskall:

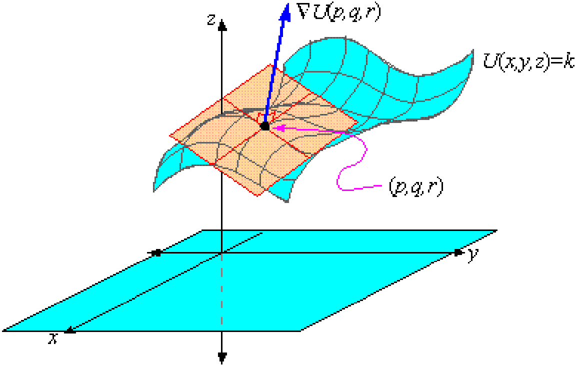
$$r = 1$$

Uavhengig av $0 \leq \theta \leq 2\pi$ og $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$.



Setning

Anta at $U : A \to \mathbb{R}$ *er en funksjon i* n *variable og at* U *er deriverbar i punktet* $\mathbf{a}$. *Dersom* $U(\mathbf{a}) = c$, *står gradienten* $\nabla U(\mathbf{a})$ *alltid normalt på nivåmengden* N_c.



Definisjon

Anta at $f : A \to \mathbb{R}$ *er en funksjon i to variable og at* f *er deriverbar i punktet* (a_0, y_o)*.* **Tangentplanet** *til* f *i punktet* $(x_0, y_0, f(x_0, y_0))$ *er definert ved likningen*

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Samme som likningen til lineariseringen av f i punktet (x_0, y_0).

Normalretningen i punktet er gitt ved

$$\mathbf{n} = -\frac{\partial f}{\partial x}(x_0, y_0)\mathbf{i} - \frac{\partial f}{\partial y}(x_0, y_0)\mathbf{j} + \mathbf{k}$$

Eksempel

$$f(x,y) = x^3y^2, \quad (x_0, y_0) = (2, -1)$$

Vi har

$$\frac{\partial f}{\partial x} = 3x^2y^2, \quad \frac{\partial f}{\partial y} = 2x^3y$$

Innsatt i punktet $(2, -1)$ *gir dette*

$$\frac{\partial f}{\partial x}(2,-1) = 12, \quad \frac{\partial f}{\partial y}(2,-1) = -16$$

som gir normalretning

$$\mathbf{n} = -12\mathbf{i} + 16\mathbf{j} + \mathbf{k}$$

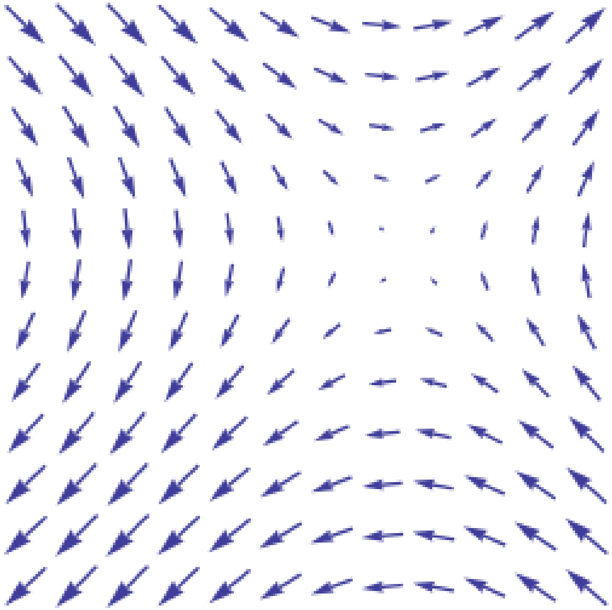
og tangentplan

$$z = f(2,-1) + 12(x-2) - 16(y+1) = 8 + 12(x-2) - 16(y+1)$$

eller

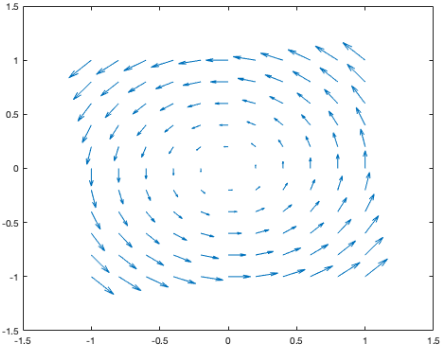
$$12x - 16y - z = 32$$

Vektorfelt



MATLAB:

$$[x, y] = meshgrid(-1 : 0.2 : 1, -1 : 0.2 : 1);$$
$$u = -y;$$
$$v = x;$$
$$quiver(x, y, u, v)$$

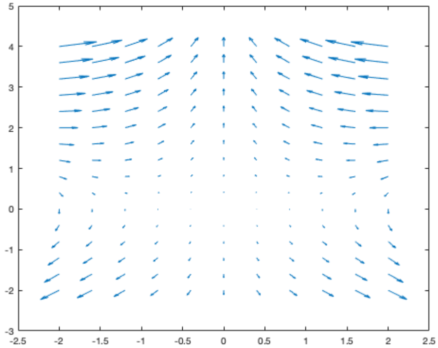


Et **kritisk punkt** for et vektorfelt **F** er et punkt $\mathbf{a} \in \mathbb{R}^n$ hvor $\mathbf{F}(\mathbf{a}) = 0$.

Eksempel

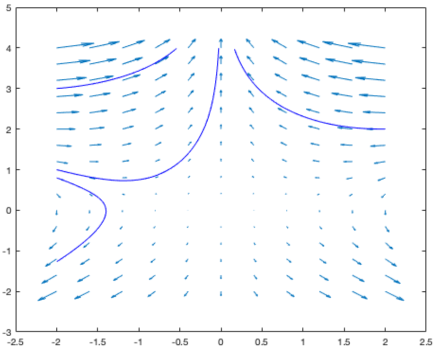
$$\mathbf{F}(x, y) = (-2xy, 2y - x^2)$$

har et kritisk punkt i origo.

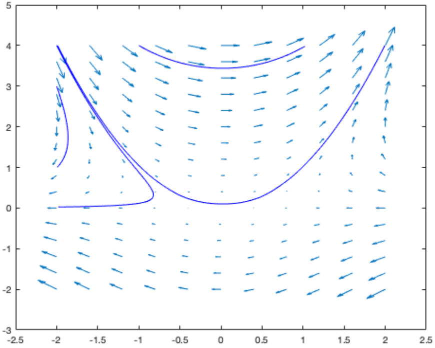


Strømningskurver

$$\mathbf{F}(x, y) = (-2xy, 2y - x^2)$$

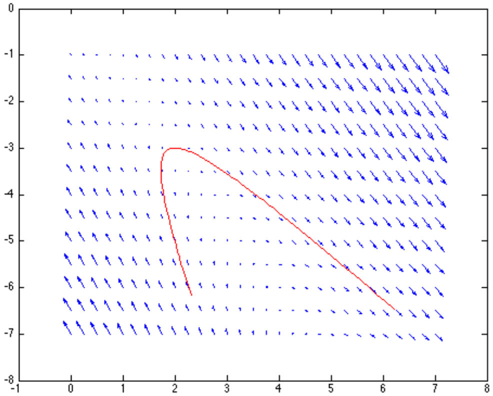


$$\mathbf{F}(x, y) = (2y - x^2, 2xy)$$



$$\mathbf{F}(x,y) = (2x + y, -3x - 2y)$$

$$\mathbf{r}(t) = (\tfrac{1}{2}e^{t} + \tfrac{3}{2}e^{-t}, -\tfrac{1}{2}e^{t} - \tfrac{9}{2}e^{-t})$$



Vektorfelt med strømningskurve

MATLAB-koden til illustrasjonen over er som følger:

```
[x,y]=meshgrid(0:0.25:7,-7:0.5:-1);
u = 2*x + y;
v = -3*x - 2*y;
quiver(x,y,u,v);
hold on
t=linspace(-0.25,2.5,100);
x1=(1/2)*exp(t)+(3/2)*exp(-t);
x2=-(1/2)*exp(t)-(9/2)*exp(-t);
plot(x1,x2,'r')
```

$$\mathbf{r}(t) = (x(t), y(t)) = (\tfrac{1}{2}e^t + \tfrac{3}{2}e^{-t}, -\tfrac{1}{2}e^t - \tfrac{9}{2}e^{-t})$$

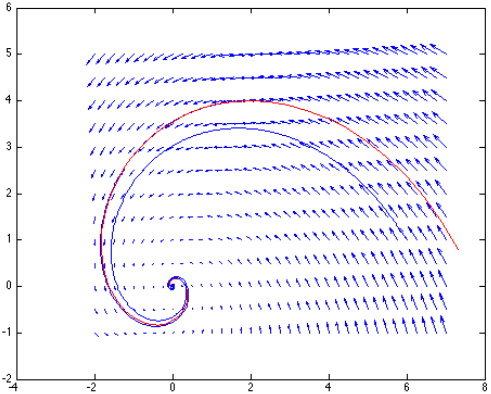
Det gir

$$x'(t) = \tfrac{1}{2}e^t - \tfrac{3}{2}e^{-t} = 2(\tfrac{1}{2}e^t + \tfrac{3}{2}e^{-t}) + (-\tfrac{1}{2}e^t - \tfrac{9}{2}e^{-t}) = 2x(t) + y(t)$$

$$y'(t) = -\tfrac{1}{2}e^t + \tfrac{9}{2}e^{-t}) = -3(\tfrac{1}{2}e^t + \tfrac{3}{2}e^{-t}) - 2(-\tfrac{1}{2}e^t - \tfrac{9}{2}e^{-t}) = -3x(t) - 2y(t)$$

$$\mathbf{F}(x,y) = (-x-2y, 2x-y)$$

$$\mathbf{r}(t) = \left(2e^{-t}\left(\cos(2t) - 2\sin(2t)\right), 2e^{-t}\left(2\cos(2t) + \sin(2t)\right)\right)$$



Vektorfelt med strømningskurve

MATLAB-koden til illustrasjonen over er som følger:

```
[x,y]=meshgrid(-2:0.25:7,-1:0.5:5);
u = -x -2*y;
v = 2*x -y;
quiver(x,y,u,v);
hold on
t=linspace(-0.5,4,100);
x1=exp(-t).*(2*cos(2*t)-4*sin(2*t));
x2=exp(-t).*(4*cos(2*t)+2*sin(2*t));
plot(x1,x2,'r')
streamline(x,y,u,v,6,1);
```


Briggs-Rauscher Reaction
The Briggs-Rauscher reaction is an oscillating clock reaction that changes color from colorless to amber to dark blue and then repeats the colors changes about ten times.
sciencenotes.org

Lotka-Volterra-likningen.

Vi betrakter et koblet system av to autokatalytiske reaksjoner der konsentrasjonen av en av reaktantene A er mye høyere en likevektsverdien. Dermed kan vi neglisjere den reverserte reaksjonen og bare konsentrere oss om en retning. Det idealiserte systemet ser ut som

$$\begin{aligned} A + X &\to 2X \\ X + Y &\to 2Y \\ Y &\to E \end{aligned}$$

og den matematiske modellen er gitt ved

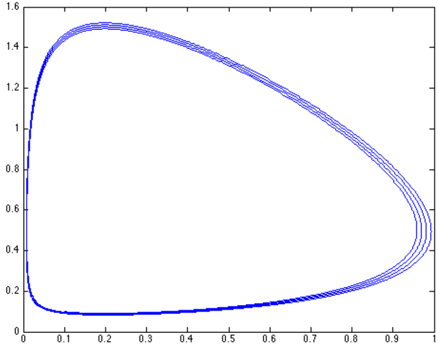
$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A][X] \\ \frac{d[X]}{dt} &= k_1[A][X] - k_2[X][Y] \\ \frac{d[Y]}{dt} &= k_2[X][Y] - k_3[Y] \end{aligned}$$

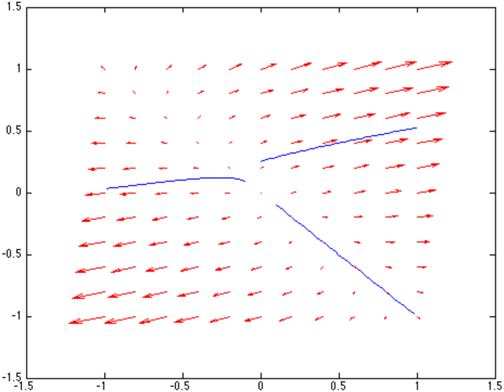
der $[A]$ betyr konsentrasjonen av stoffet A.

Dette gir vektorfelt (med tallverdier for konstantene k_1, k_2 og k_3)

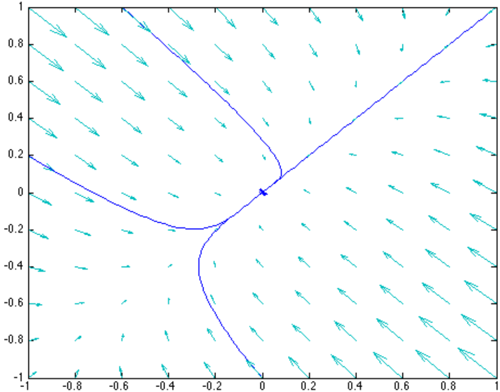
$$\mathbf{LV}([A],[X],[Y]) = (-0.001[A][X], 0.001[A][X]-[X][Y], [X][Y]-0.2[Y])$$

Strømningslinjene til dette feltet i $([X],[Y])$-planet ($[A]$ er veldig nær å være konstant)

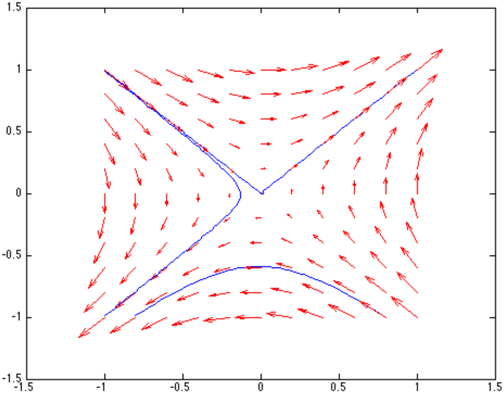




Repellor (frastøter)

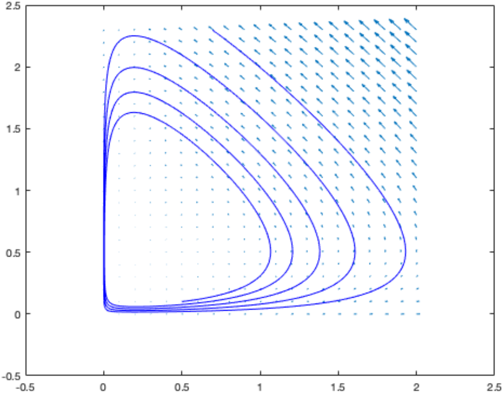


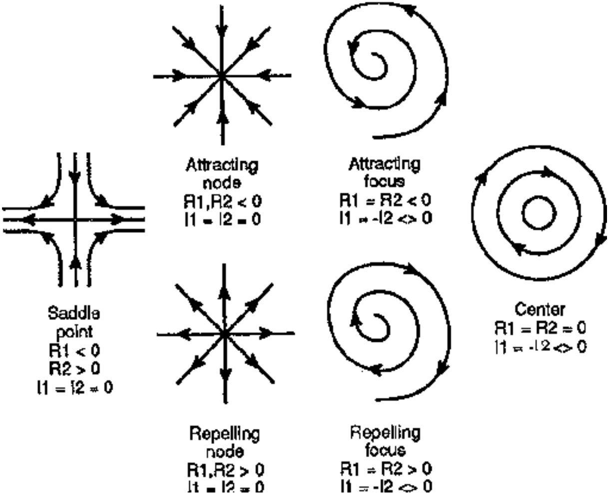
Attraktor (tiltrekker)



Bifurkasjon

$$(\frac{1}{2}x - xy, xy - 0.2y)$$




Saddle point
R1 < 0
R2 > 0
I1 = I2 = 0
Attracting node
R1,R2 < 0
I1 = I2 = 0
Repelling node
R1,R2 > 0
I1 = I2 = 0
Attracting focus
R1 = R2 < 0
I1 = -I2 <> 0
Repelling focus
R1 = R2 > 0
I1 = -I2 <> 0
Center
R1 = R2 = 0
I1 = -I2 <> 0