# MAT 1110, 11. mars 2022

* Integral over flater
* Variabelskifte
* Uegentlige integral

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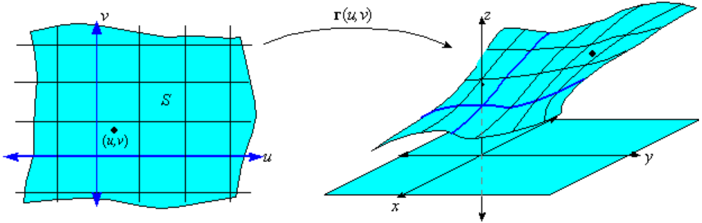
# Integral over flater

Gitt parametrisert flate $S$

$$\mathbf{r} : A \to \mathbb{R}^3$$




## Definisjon

*La $A$ være et område i $(x,y)$-planet og $\mathbf{r} : A \to \mathbb{R}^3$ en parametrisert flate som vi kaller $S$. Da har vi*

$$\mathit{areal}(S) = \iint_S 1\, dS = \iint_A |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

## Definisjon

*La $A$ være et område i $(x, y)$-planet og $\mathbf{r} : A \to \mathbb{R}^3$ en parametrisert flate som vi kaller $S$. La $f$ være en kontinuerlig funksjon på $S$. Da kaller vi*

$$\iint_S f\, dS = \iint_A f(\mathbf{r}(u, v))\,|\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

*for* **flateintegralet** *av $f$ over $S$.*

## Eksempel

$$f(x,y,z) = xyz^2$$

$$\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u^2\,\mathbf{k}, \quad 0 \le u \le 2, \quad 0 \le v \le 2\pi$$

$$\frac{\partial \mathbf{r}}{\partial u} = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + 2u\,\mathbf{k} \qquad \frac{\partial \mathbf{r}}{\partial v} = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = -2u^2\cos v\,\mathbf{i} - 2u^2\cos v\,\mathbf{j} + u\,\mathbf{k}$$

$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| = |-2u^2\cos v\,\mathbf{i} - 2u^2\cos v\,\mathbf{j} + u\,\mathbf{k}| = u\sqrt{4u^2+1}$$

$$\iint_S f\,dS = \int_0^{2\pi}\int_0^2 f(\mathbf{r}(u,v))\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\,dv = 0$$

# Variabelskifte i integral

## Setning

*La* $U \subset \mathbb{R}^2$ *være en åpen, begrenset mengde og la* $\mathbf{T} : U \to \mathbb{R}^2$ *være en injektiv funksjon med kontinuerlige partiellderiverte slik at* $\det \mathbf{T} \neq 0$ *på hele* $U$*. Hvis* $D \subset U$ *er en lukket, Jordan-målbar mengde og* $f : \mathbf{T}(D) \to \mathbb{R}$ *er en kontinuerlig funksjon, så er*

$$\iint_A f(x,y)\,dx\,dy = \iint_D f(\mathbf{T}(u,v))|\mathbf{T}'(u,v)|\,du\,dv$$

*der* $A = \mathbf{T}(D)$, $\mathbf{T}(u,v) = (x(u,v), y(u,v))$ *og*

$$\mathbf{T}'(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Vi bruker ofte notasjonen

$$\mathbf{T}'(u,v) = \frac{\partial(x,y)}{\partial(u,v)}$$

Vi kan betrakte $\mathbf{T} : U \to \mathbb{R}^2$ som en parametrisert flate i $\mathbb{R}^3$ ved

$$\mathbf{T}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + 0 \cdot \mathbf{k}$$

Det gir

$$\mathbf{T}_u(u,v) = \frac{\partial x}{\partial u}(u,v)\,\mathbf{i} + \frac{\partial y}{\partial u}(u,v)\,\mathbf{j} \quad \mathbf{T}_v(u,v) = \frac{\partial x}{\partial v}(u,v)\,\mathbf{i} + \frac{\partial y}{\partial v}(u,v)\,\mathbf{j}$$

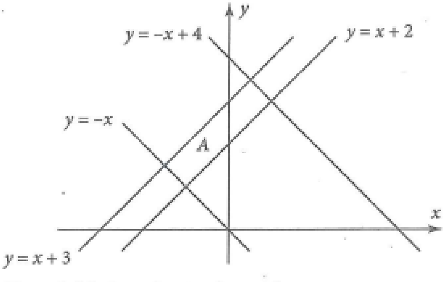
og

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

og videre

$$|\mathbf{T}_u \times \mathbf{T}_v| = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = |\mathbf{T}'(u,v)|$$

# Eksempel




$$2 \leq y - x \leq 3, \qquad 0 \leq y + x \leq 4$$

*Substituerer*

$$x = \tfrac{1}{2}(-u + v) \qquad y = \tfrac{1}{2}(u + v)$$

*som gir*

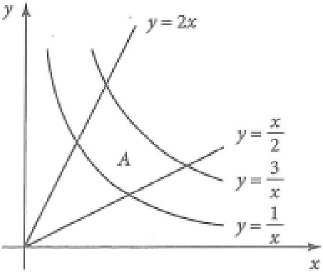
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

## Eksempel

*Vi skal integrere funksjonen $(x, y) = xy$ over området $A$.*

$$\begin{aligned}
\iint_A xy\,dx\,dy &= \int_2^3 \int_0^4 \Big(\frac{-u+v}{2}\Big)\Big(\frac{+u+v}{2}\Big)\Big|-\tfrac{1}{2}\Big|\,du\,dv \\
&= \frac{1}{8}\int_2^3 \int_0^4 v^2 - u^2\,dv\,du \\
&= \frac{1}{8}\int_2^3 [\tfrac{1}{3}v^3 - u^2 v]_0^4\,dv\,du \\
&= \frac{1}{8}\int_2^3 \frac{64}{3} - 4u^2\,du \\
&= \frac{1}{8}[\tfrac{64}{3}u - \tfrac{4}{3}u^3]_2^3 = -\frac{1}{2}
\end{aligned}$$

# Eksempel




*Vi skal integrere* $f(x,y) = \frac{x}{y}$ *over området A gitt på figuren.*

$$1 \leq xy \leq 3, \qquad \frac{1}{2} \leq \frac{y}{x} \leq 2$$

*Setter* $u = xy$ *og* $v = \frac{y}{x}$, *som gir* $x = \sqrt{\frac{u}{v}}$ *og* $y = \sqrt{uv}$ *og hvor* $1 \leq u \leq 3$, $\frac{1}{2} \leq v \leq 2$.

# Eksempel

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2\sqrt{v^3}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{2v}$$

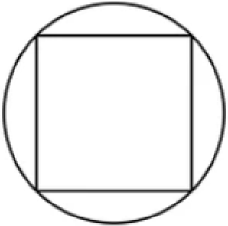
$$\begin{aligned} \iint_A f\,dA &= \iint \frac{x}{y}\,dx\,dy \\ &= \iint_D \frac{\sqrt{\frac{u}{v}}}{\sqrt{uv}} \frac{1}{2v}\,du\,dv \\ &= \int_1^3 \int_{\frac{1}{2}}^2 \frac{1}{2v^2}\,dv\,du \\ &= \int_{\frac{1}{2}}^2 \frac{1}{v^2}\,dv \\ &= [-\frac{1}{v}]_{\frac{1}{2}}^2 = \frac{3}{2} \end{aligned}$$

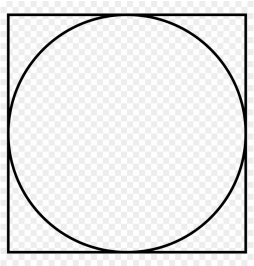
# Uegentlige integraler

$$K_n = \{(x,y) \in \mathbb{R}^2 \mid |x|, |y| \leq n\}$$

$$B(\mathbf{0}, n) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq n^2\}$$

$$K_{\frac{n}{\sqrt{2}}} \subset B(\mathbf{0}, n) \subset K_n \subset B(\mathbf{0}, \sqrt{2}n)$$





## Definisjon

*La* $A \subset \mathbb{R}^2$ *slik at* $A \cap K_n$ *er Jordan-målbar for alle* $n \in \mathbb{N}$, *og la* $f : A \to \mathbb{R}$ *være en ikke-negativ, kontinuerlig funksjon. Vi definerer*

$$\iint_A f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap K_n} f(x,y)\,dx\,dy$$

*hvis grensen eksisterer. Vi sier da at det* **uegentlige integralet** $\iint_A f(x,y)\,dx\,dy$ **konvergerer**. *Dersom det ikke konvergerer sier vi at det* **divergerer**.

## Eksempel

$$f(x,y) = \frac{y^2}{1+x^2}, \qquad \mathbb{R} \times [0,2]$$

$$\begin{aligned}
\iint_A f(x,y)\,dx\,dy &= \lim_{n\to\infty} \int_{-n}^{n} \int_0^2 \frac{y^2}{1+x^2}\,dy\,dx \\
&= \lim_{n\to\infty} \int_{-n}^{n} [\frac{1}{3}\frac{y^3}{1+x^2}]_0^2\,dx \\
&= \lim_{n\to\infty} \frac{8}{3} \int_{-n}^{n} \frac{1}{1+x^2}\,dx \\
&= \lim_{n\to\infty} \frac{8}{3}(\arctan(n) - \arctan(-n)) \\
&\to \frac{8}{3}(\frac{\pi}{2} - (-\frac{\pi}{2})) = \frac{8\pi}{3}
\end{aligned}$$

## Setning

*La* $A \subset \mathbb{R}^2$ *slik at* $A \cap B(\mathbf{0}, n)$ *er Jordan-målbar for alle* $n \in \mathbb{N}$*, og la* $f : A \to \mathbb{R}$ *være en ikke-negativ, kontinuerlig funksjon. Da har vi*

$$\iint_A f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap B(\mathbf{0},n)} f(x,y)\,dx\,dy$$

# Eksempel

$$f(x,y) = e^{-\frac{x^2+y^2}{2}}, \qquad (x,y) \in \mathbb{R}^2$$

*Vi setter*

$$I_n = \int_{-n}^{n} e^{-\frac{x^2}{2}}$$

$$\begin{aligned}
\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy &= \lim_{n\to\infty} \int_{-n}^{n}\int_{-n}^{n} e^{-\frac{x^2+y^2}{2}}\,dy\,dx \\
&= \lim_{n\to\infty} \int_{-n}^{n} e^{-\frac{y^2}{2}} \left( \int_{-n}^{n} e^{-\frac{x^2}{2}}\,dx \right) dy \\
&= I_n \lim_{n\to\infty} \int_{-n}^{n} e^{-\frac{y^2}{2}}\,dy = (I_n)^2
\end{aligned}$$

# Eksempel

$$f(x,y) = e^{-\frac{x^2+y^2}{2}}, \qquad (x,y) \in \mathbb{R}^2$$

*Vi bruker polarkoordinater*

$$\begin{aligned}
\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy &= \lim_{n\to\infty} \iint_{B(\mathbf{0},n)} f(x,y)\,dx\,dy \\
&= \lim_{n\to\infty} \int_0^n \int_0^{2\pi} e^{-\frac{r^2}{2}}\, r\,d\theta\,dr \\
&= 2\pi \lim_{n\to\infty} \int_0^n e^{-\frac{r^2}{2}}\, r\,dr \\
&= 2\pi \lim_{n\to\infty} \int_0^{\frac{n^2}{2}} e^{-u}\,du \\
&= 2\pi \lim_{n\to\infty} (1 - e^{-\frac{n^2}{2}}) = 2\pi
\end{aligned}$$

*Det følger at*

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = \sqrt{2\pi} \qquad \Rightarrow \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = 1$$

$$f_+(\mathbf{x}) = \begin{cases} f(x) & \text{hvis } f(\mathbf{x}) > 0 \\ 0 & \text{ellers} \end{cases} \qquad f_-(\mathbf{x}) = \begin{cases} -f(x) & \text{hvis } f(\mathbf{x}) < 0 \\ 0 & \text{ellers} \end{cases}$$

## Definisjon

*La $A \subset \mathbb{R}^2$ slik at $A \cap K_n$ er Jordan-målbar for alle $n \in \mathbb{N}$, og la $f : A \to \mathbb{R}$ være en begrenset, kontinuerlig funksjon. Vi sier at integralet $\iint_A f(x,y)\,dx\,dy$* **konvergerer** *dersom begge integralene $\iint_A f_+(x,y)\,dx\,dy$ og $\iint_A f_-(x,y)\,dx\,dy$* **konvergerer**. *I så fall setter vi*

$$\iint_A f(x,y)\,dx\,dy = \iint_A f_+(x,y)\,dx\,dy - \iint_A f_-(x,y)\,dx\,dy$$

Vi merker oss at begge de to integralene må konvergerer, det holder ikke at de konvergerer sammen, f.eks. konvergerer ikke

$$\lim_{n\to\infty}\int_{-n}^{n} x\,dx$$

selv om

$$\int_{-n}^{n} x\,dx = [\frac{1}{2}x^2]_{-n}^{n} = \frac{1}{2}n^2 - \frac{1}{2}n^2 = 0 \to 0 \quad \text{når } n \to \infty$$

# Eksempel

$$f(x,y) = \frac{1}{(x^2+y^2)^p} \qquad (x,y) \in B(\mathbf{0},1), \quad p > 0$$

*Siden funksjonen ikke er definert i 0, snevrer vi inn området til*

$$A_\varepsilon = \{\mathbf{x} \in \mathbb{R}^2 \,|\, \varepsilon \leq |\mathbf{x}| \leq 1\}$$

*Vi bruker polarkoordinater*

$$\begin{aligned}
\iint_{B(\mathbf{0},1)} f(x,y)\,dx\,dy &= \int_\varepsilon^1 \int_0^{2\pi} \frac{1}{r^{2p}}\, r\, d\theta\, dr \\
&= 2\pi \int_\varepsilon^1 \frac{1}{r^{2p}}\, r\, dr \\
&= 2\pi \int_\varepsilon^1 r^{1-2p}\, dr \\
&= \frac{2\pi}{2-2p}(1-\varepsilon^{2-2p}) \\
&= \frac{\pi}{1-p}(1-\varepsilon^{2-2p}) \to \frac{\pi}{1-p} \quad \text{når} \quad \varepsilon \to 0, \quad p \neq 1
\end{aligned}$$

## Eksempel

$$\iint_{A_\varepsilon} \frac{1}{x^2+y^2}\,dx\,dy = \int_\varepsilon^1 \int_0^{2\pi} \frac{1}{r^2}\, r\, d\theta\, dr$$

$$= 2\pi \int_\varepsilon^1 r^{-1}\, dr = -2\pi \ln \varepsilon \to \infty \quad \textit{når } \varepsilon \to 0$$

# Pensum, midtveiseksamen:

1.9, 1.10
2.7, 2.8
3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9
4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10, 4.12
6.1, 6.2, 6.3, 6.4