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∗ Trippelintegral

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Trippelintegral

Rektangulær boks: $R = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$ og funksjon $f : R \to \mathbb{R}$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_{a_1}^{a_2} \left(\int_{b_1}^{b_2} \left(\int_{c_1}^{c_2} f(x,y,z)\,dz \right) dy \right) dx$$

f kontinuerlig $\Rightarrow$ f integrerbar

Eksempel

Vi skal regne ut $\iiint_R (x + ye^{2z})\,dx\,dy\,dz$ *over området* $R = [0,1] \times [1,3] \times [0,2]$

$$
\begin{aligned}
\iiint_R (x + ye^{2z})\,dx\,dy\,dz &= \int_0^1 \left(\int_1^3 \left(\int_0^2 (x + ye^{2z})\,dz \right) dy \right) dx \\
&= \int_0^1 \left(\int_1^3 (2x + \frac{1}{2} y(e^4 - 1))\,dy \right) dx \\
&= \int_0^1 (2x(3-1) + \frac{1}{4}(3^2 - 1^2)(e^4 - 1))\,dx \\
&= \int_0^1 (4x + 2(e^4 - 1))\,dx \\
&= 2 + 2(e^4 - 1)) = 2e^4
\end{aligned}
$$

Eksempel

Vi skal regne ut $\iiint_R x\,dx\,dy\,dz$ *over området* S*, som ligger over* (x,y)*-planet og under paraboloiden* $z = 4 - x^2 - y^2$*. Vi lar* D *være disken gitt ved* $x^2 + y^2 \leq 4$ *i* (x,y)*-planet.*

$$\begin{aligned}\iiint_S x\,dx\,dy\,dz &= \iint_D \left(\int_0^{4-x^2-y^2} x\,dz \right) dx\,dy \\ &= \iint_D 4x - x^3 - xy^2\,dx\,dy\end{aligned}$$

Skifter til polarkoordinater:

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2$$

$$\begin{aligned}\iiint_S x\,dx\,dy\,dz &= \iint_D 4x - x^3 - xy^2\,dx\,dy \\ &= \int_0^{2\pi} \left(\int_0^2 (4r\cos\theta - r^3\cos^3\theta - r^3\cos\theta\sin^2\theta) r\,dr \right) d\theta\end{aligned}$$

Eksempel

$$\begin{aligned}
\iiint_S x\,dx\,dy\,dz &= \iint_D 4x - x^3 - xy^2\,dx\,dy \\
&= \int_0^{2\pi}\left(\int_0^2 (4r\cos\theta - r^3\cos^3\theta - r^3\cos\theta\sin^2\theta)r\,dr\right)\,d\theta \\
&= \int_0^{2\pi} [\frac{4}{3}r^3\cos\theta - \frac{1}{5}r^5\cos^3\theta - \frac{1}{5}r^5\cos\theta\sin^2\theta]_0^2\ \ d\theta \\
&= \int_0^{2\pi}\left(\frac{32}{3}\cos\theta - \frac{32}{5}\cos^3\theta - \frac{32}{5}\cos\theta\sin^2\theta\right)\,d\theta \\
&= \int_0^{2\pi}\left(\frac{32}{3}\cos\theta - \frac{32}{5}\cos\theta\right)\,d\theta = 0
\end{aligned}$$

Eksempel

Vi skal integrere funksjonen $f(x,y,z) = xy$ *over området* S *som ligger mellom planet* $z = 2x + 4y$ *og paraboloiden* $z = x^2 + y^2$

Skjæringen mellom planet og paraboloiden er gitt ved

$$2x + 4y = x^2 + y^2 \quad \Rightarrow \quad x^2 - 2x + y^2 - 4y = (x-1)^2 + (y-2)^2 - 5$$

Vi lar A *være området gitt ved* $(x-1)^2 + (y-2)^2 \leq 5$ *i* (x,y)*-planet. Det gir*

$$\iiint_S xy\,dx\,dy\,dz = \iint_A \left(\int_{x^2+y^2}^{2x+4y} xy\,dz \right) dx\,dy$$

og

$$\int_{x^2+y^2}^{2x+4y} xy\,dz = xy(2x + 4y - x^2 - y^2)$$

Eksempel

Vi skifter til polarkoordinater

$$x = 1 + r\cos\theta, \quad y = 2 + r\sin\theta$$

hvor $0 \le \theta \le 2\pi$ *og* $0 \le r \le \sqrt{5}$. *Det gir*

$$\begin{aligned}
\iiint_S xy\,dx\,dy\,dz &= \iint_A \left(xy(2x + 4y - x^2 - y^2)\right)\,dx\,dy \\
&= \iint_A \left(xy(5 - (x-1)^2 - (y-2)^2\right)\,dx\,dy \\
&= \int_0^{\sqrt{5}} \left(\int_0^{2\pi} (1 + r\cos\theta)(2 + r\sin\theta)(5 - r^2)r\,d\theta\right)\,dr \\
&= \int_0^{\sqrt{5}} (5 - r^2)r \left(\int_0^{2\pi} (2 + 2r\cos\theta + r\sin\theta + r^2\cos\theta\sin\theta)\,d\theta\right)\,dr \\
&= 4\pi \int_0^{\sqrt{5}} (5r - r^3)\,dr \\
&= 4\pi[\frac{5}{2}r^2 - \frac{1}{4}r^4]_0^{\sqrt{5}} = 4\pi\left(\frac{25}{2} - \frac{25}{4}\right) = 25\pi
\end{aligned}$$

$$1_V(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in V \\ 0 & \mathbf{x} \notin V \end{cases}$$

$$\iiint_V dV = \iiint_{\mathbb{R}^3} 1_V \, dV$$

Definisjon

En mengde $V \subset \mathbb{R}^3$ er **Jordan-målbar** *dersom 1_V er integrerbar.*

Setning

La $U \subset \mathbb{R}^3$ være en åpen, begrenset mengde og la $\mathbf{T} : U \to \mathbb{R}^3$ være en injektiv funksjon med kontinuerlige partiellderiverte slik at $\det \mathbf{T} \neq 0$ på hele U. Hvis $D \subset U$ er en lukket, Jordan-målbar mengde og $f : \mathbf{T}(D) \to \mathbb{R}$ er en kontinuerlig funksjon, så er

$$\iiint_A f(x,y)\,dx\,dy\,dz = \iiint_D f(\mathbf{T}(u,v,w))|\mathbf{T}'(u,v,w)|\,du\,dv\,dw$$

der $A = \mathbf{T}(D)$, $\mathbf{T}(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ og

$$\mathbf{T}'(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Vi bruker ofte notasjonen

$$\mathbf{T}'(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

A: parallellepipedet utspent av $\mathbf{a} = (1, 0, -1)$, $\mathbf{b} = (0, 1, 2)$, $\mathbf{c} = (2, 3, 1)$, funksjon $f(x, y, z) = 4x - y + 2z$.

$$\mathbf{T}(u, v, w) = (u + 2w, v + 3w, -u + 2v + w)$$

$$\mathbf{T}'(u, v, w) = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix} = -3$$

$$\begin{aligned} &\iiint_A (4x - y + 2z)\, dx\, dy\, dz \\ &\quad = \iiint_D (4(u + 2w) - (v + 3w) + 2(-u + 2v + w) \cdot |-3|\, du\, dv\, dw \\ &\quad = \iiint_D (6u + 9v + 21w)\, du\, dv\, dw \\ &\quad = \int_0^1 \int_0^1 \int_0^1 (6u + 9v + 21w)\, du\, dv\, dw \end{aligned}$$

$$
\begin{aligned}
\iiint_A (4x - y + 2z)\,dx\,dy\,dz &= \int_0^1 \int_0^1 \int_0^1 (6u + 9v + 21w)\,du\,dv\,dw \\
&= \int_0^1 \int_0^1 [3u^2 + 9uv + 21uw]_0^1\,dv\,dw \\
&= \int_0^1 \int_0^1 (3 + 9v + 21w)\,dv\,dw \\
&= \int_0^1 (3 + \frac{9}{2} + 21w)\,dw \\
&= 3 + \frac{9}{2} + \frac{21}{2} = 18
\end{aligned}
$$

Sylinderkoordinater:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z$$

med Jacobi-determinant:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_D f(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz$$

Eksempel

$$f(x,y,z) = x^2 z$$

A: over (x,y)-planet, inni sylinderen $x^2+y^2=1$ og kula $x^2+y^2+z^2=4$.

$$I = \iiint_A x^2 z \, dx \, dy \, dz = \iiint_D (r\cos\theta)^2 \, z \, r \, dr \, d\theta \, dz$$

D: $0 \le \theta \le 2\pi,\; 0 \le r \le 1,\; 0 \le z \le \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r^3\cos^2\theta \, z \, dz \, dr \, d\theta \\
&= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta \, \frac{1}{2}(4-r^2) \, dr \, d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\cos^2\theta \left(\int_0^1 r^3(4-r^2)\, dr\right) d\theta \\
&= \frac{5}{12}\int_0^{2\pi}\cos^2\theta \, d\theta = \frac{5}{12}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right) d\theta = \frac{5\pi}{12}
\end{aligned}$$

Kulekoordinater:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

med Jacobi-determinant:

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2 \sin\phi$$

$$\iiint_A f(x,y,z)\,dx\,dy\,dz$$
$$= \iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

Gjennomsnittsavstanden fra et punkt $P = (0, 0, a)$ til en kule K med radius 1 og sentrum i origo. Avstanden fra P til et punkt (x, y, z) inne i kula er gitt ved:

$$\sqrt{x^2 + y^2 + (z-a)^2}$$

Gjennomsnittsavstanden:

$$\overline{a} = \frac{1}{\text{volum}(K)} \iiint_K \left(\sqrt{x^2 + y^2 + (z-a)^2} \right) dx\, dy\, dz$$

Med kulekoordinater:

$$(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 + (\rho \cos\phi - a)^2 = \rho^2 - 2a\rho\cos\phi + a^2$$

$$\overline{a} = \frac{1}{\text{volum}(K)} \iiint_K \left(\sqrt{\rho^2 - 2a\rho\cos\phi + a^2} \right) \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$\overline{a} = \frac{3}{4\pi}\int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^1\left(\sqrt{\rho^2 - 2a\rho\cos\phi + a^2}\right)\rho^2\sin\phi\,d\rho\right)d\phi\right)d\theta$$
$$= \frac{3}{2}\int_0^1\left(\int_0^{\pi}\left(\sqrt{\rho^2 - 2a\rho\cos\phi + a^2}\right)\rho^2\sin\phi\,d\phi\right)d\rho$$

Setter $u = \rho^2 - 2a\rho\cos\phi + a^2$ som gir $du = 2a\rho\sin\phi\,d\phi$

$$\overline{a} = \frac{3}{2}\int_0^1\left(\int_{(\rho-a)^2}^{(\rho+a)^2}\left(\frac{\rho}{2a}\sqrt{u}\right)du\right)d\rho$$
$$= \int_0^1\left((\rho+a)^3 + (\rho-a)^3\right)\frac{\rho}{2a}\,d\rho$$
$$= \int_0^1\left(2\rho^3 + 6a^2\rho\right)\frac{\rho}{2a}\,d\rho$$
$$= \int_0^1\left(\frac{1}{a}\rho^4 + 3a\rho^2\right)d\rho$$
$$= \frac{1}{5a} + a$$

La S være et legeme i $\mathbb{R}^3$ med varierende tetthet, gitt ved $f(x,y,z)$. Da vil den totale massen av legemet være gitt ved

$$M = \iiint_S f(x,y,z)\,dx\,dy\,dz$$

Dersom tettheten overalt er 1, får vi ut volumet;

$$\text{volum}(S) = \iiint_S 1\,dx\,dy\,dz$$

Volumet av en kule i kulekoordinater:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

hvor $0 \le \rho \le R$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$ med Jacobi-determinant:

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin\phi$$

Det gir

$$\begin{aligned}
\text{volum} &= \int_0^R \left(\int_0^\pi \left(\int_0^{2\pi} \rho^2 \sin\phi \, d\theta \right) d\phi \right) d\rho \\
&= 2\pi \int_0^R \rho^2 \left(\int_0^\pi \sin\phi \, d\phi \right) d\rho \\
&= 4\pi \int_0^R \rho^2 \, d\rho = \frac{4}{3}\pi R^3
\end{aligned}$$

Vi skal beregne gravitasjonen inne i en hul kule S, dvs. $r \leq \rho \leq R$. Vi betrakter et punkt $P = (0, 0, c)$ inne i hulrommet. Avstanden fra P til et punkt på kuleskallet er gitt ved

$$\begin{aligned}
&\sqrt{(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 + (c - \rho\cos\phi)^2} \\
&= \sqrt{\rho^2 \sin^2\phi + (c - \rho\cos\phi)^2} \\
&= \sqrt{c^2 - 2c\rho\cos\phi + \rho^2}
\end{aligned}$$

Gravitasjonskraften er gitt ved $\frac{1}{\text{avstand}^2}$, så samlet kraft på punktet P blir

$$\frac{(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, c - \rho\cos\phi)}{(c^2 - 2c\rho\cos\phi + \rho^2)^{\frac{3}{2}}}$$

Ved en symmetribetraktning ser vi at samlet kraft i x- og y-retningene blir 0, så vi trenger bare å regne ut

$$\int_r^R \left(\int_0^\pi \left(\int_0^{2\pi} \frac{(c - \rho\cos\phi)\rho^2 \sin\phi}{(c^2 - 2c\rho\cos\phi + \rho^2)^{\frac{3}{2}}} \, d\theta \right) d\phi \right) d\rho$$

Vi setter $u = c^2 - 2c\rho\cos\phi + \rho^2$ og $du = 2c\rho\sin\phi\, d\phi$ og vi får $c - \rho\cos\phi = \frac{u+c^2-\rho^2}{2c}$

$$\int_r^R \left(\int_0^\pi \left(\int_0^{2\pi} \frac{(c-\rho\cos\phi)\rho^2\sin\phi}{(c^2-2c\rho\cos\phi+\rho^2)^{\frac{3}{2}}}\, d\theta \right) d\phi \right) d\rho$$

$$= 2\pi \int_r^R \left(\int_{(\rho-c)^2}^{(\rho+c)^2} \frac{u+c^2-\rho^2}{2c} \frac{\rho}{2c} \frac{du}{(u)^{\frac{3}{2}}} \right) d\rho$$

$$= \frac{\pi}{2c^2} \int_r^R \rho \left(\int_{(\rho-c)^2}^{(\rho+c)^2} \frac{du}{\sqrt{u}} \right) d\rho + \frac{\pi}{2c^2} \int_r^R (c^2-\rho^2)\rho \left(\int_{(\rho-c)^2}^{(\rho+c)^2} \frac{du}{(u)^{\frac{3}{2}}} \right) d\rho$$

$$= \frac{\pi}{2c^2} \int_r^R \rho[2\sqrt{u}]_{(\rho-c)^2}^{(\rho+c)^2}\, d\rho + \frac{\pi}{2c^2} \int_r^R (c^2-\rho^2)\rho[\frac{-2}{\sqrt{u}}]_{(\rho-c)^2}^{(\rho+c)^2}\, d\rho$$

$$= \frac{\pi}{2c^2} \int_r^R 4c\rho\, d\rho - \frac{\pi}{2c^2} \int_r^R 4c\rho\, d\rho = 0$$