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∗ Flateintegral av vektorfelt

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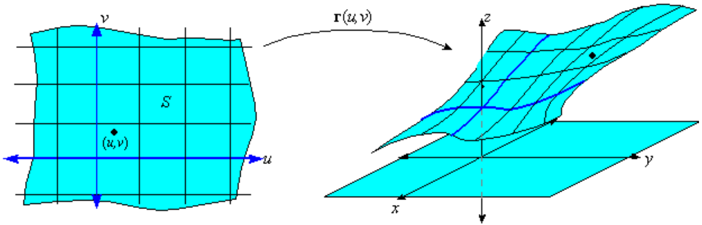
Gitt parametrisert flate S

$$\mathbf{r} : A \to \mathbb{R}^3$$



Rektangel på venstre side: $[u_i, u_{i+1}] \times [v_j, v_{j+1}]$ med areal $a_{ij} = (u_{i+1} - u_i)(v_{j+1} - v_j)$ svarer til firkant på høyre side:

$$[\mathbf{r}(u_i, v_j), \mathbf{r}(u_i, v_j) + \mathbf{r}_u(u_i, v_j)(u_{i+1} - u_i)] \times [\mathbf{r}(u_i, v_j), \mathbf{r}(u_i, v_j) + \mathbf{r}_v(u_i, v_j)(v_{j+1} - v_j)]$$

Arealet av firkanten er gitt ved

$$\begin{aligned} &|\mathbf{r}_u(u_i, v_j)(u_{i+1} - u_i) \times \mathbf{r}_v(u_i, v_j)(v_{j+1} - v_j)| \\ &\qquad = |\mathbf{r}_u(u_i, v_j) \times \mathbf{r}_v(u_i, v_j)|(u_{i+1} - u_i)(v_{j+1} - v_j) \\ &\qquad = |\mathbf{r}_u(u_i, v_j) \times \mathbf{r}_v(u_i, v_j)|a_{ij} \end{aligned}$$

Den siste likheten skriver vi

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

hvor dA svarer til et infinitesimalt rektangel i (u, v)-planet.

Definisjon

La A være et område i (x, y)-planet og $\mathbf{r} : A \to \mathbb{R}^3$ en parametrisert flate som vi kaller S. Da har vi

$$\mathit{areal}(S) = \iint_S 1 \, dS = \iint_A |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

Definisjon

La A være et område i (x, y)-planet og $\mathbf{r} : A \to \mathbb{R}^3$ en parametrisert flate som vi kaller S. La f være en kontinuerlig funksjon på S. Da kaller vi

$$\iint_S f \, dS = \iint_A f(\mathbf{r}(u, v)) \, |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

for **flateintegralet** *av f over S.*

Eksempel

$$f(x, y, z) = xyz^2$$

$$\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + u^2\, \mathbf{k}$$

$0 \leq u \leq 2,\ 0 \leq v \leq 2\pi.$

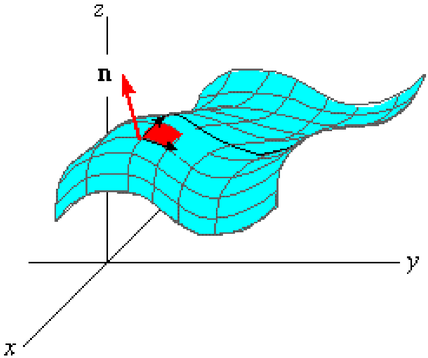
$$\frac{\partial \mathbf{r}}{\partial u} = \cos v\, \mathbf{i} + \sin v\, \mathbf{j} + 2u\, \mathbf{k} \qquad \frac{\partial \mathbf{r}}{\partial v} = -u \sin v\, \mathbf{i} + u \cos v\, \mathbf{j}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix}$$
$$= -2u^2 \cos v\, \mathbf{i} - 2u^2 \cos v\, \mathbf{j} + u\, \mathbf{k}$$

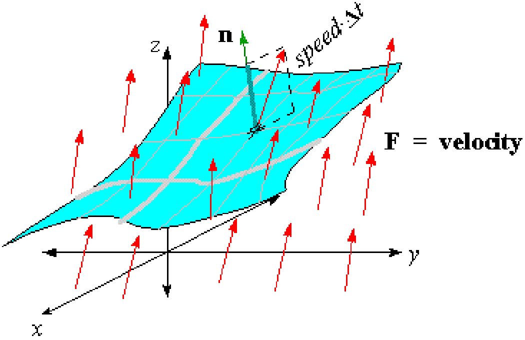
Eksempel

$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| = \left|-2u^2 \cos v\, \mathbf{i} - 2u^2 \cos v\, \mathbf{j} + u\, \mathbf{k}\right| = u\sqrt{4u^2+1}$$

$$\begin{aligned}
\iint_S f\, dS &= \int_0^{2\pi}\int_0^2 f(\mathbf{r}(u,v))\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv \\
&= \int_0^{2\pi}\int_0^2 (u^6 \sin v \cos v)(u\sqrt{4u^2+1})\, du\, dv \\
&= \int_0^{2\pi}\int_0^2 u^7\sqrt{4u^2+1} \sin v \cos v\, du\, dv \\
&= \int_0^2 u^7\sqrt{4u^2+1}\left(\int_0^{2\pi} \tfrac{1}{2}\sin(2v)\, dv\right) du = 0
\end{aligned}$$



(Enhets-) **Flatenormalen** til flaten i et punkt er en vektor **n** (av lengde 1) som står normalt på tangentplanet til flaten i punktet.



Komponenten av et vektorfelt langs en flatenormal kalles **fluksen** av vektorfeltet gjennom flaten.

Definisjon

Gitt en flate S med flatenormal $\mathbf{n}$ *og et vektorfelt* $\mathbf{F}$. **Fluksen** *av vektorfeltet gjennom flaten er punktvis gitt ved*

$$\mathbf{F} \cdot \mathbf{n}$$

og over hele flaten:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

Definisjon

La $\mathbf{r}: A \to \mathbb{R}^3$ *være en glatt parametrisert flate* T *definert på en lukket, begrenset Jordan-målbar delmengde* $A \subset \mathbb{R}^2$*. For et kontinuerlig vektorfelt* $\mathbf{F}$ *på* T *definerer vi* **flateintegralet** *til* $\mathbf{F}$ *over* T *til å være*

$$\int_T \mathbf{F} \cdot \mathbf{n} \, dS = \iint_A \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right)(u,v) \, du \, dv$$

Produktet $f = \mathbf{F} \cdot \mathbf{n}$ definerer et skalarfelt på flaten S, og vi har

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S f \, dS$$

$$\iint_S f\,dS = \iint_A f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$$

Vi setter inn $f = \mathbf{F}\cdot\mathbf{n}$ og får

$$\begin{aligned}\iint_S f\,dS &= \int_S \mathbf{F}\cdot\mathbf{n}\,dS \\ &= \iint_A \mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{n}(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv\end{aligned}$$

Vi har

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \quad \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \mathbf{n}|\mathbf{r}_u \times \mathbf{r}_v|$$

som gir

$$\iint_S f\,dS = \iint_A \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv$$

Kulekoordinater:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

med Jacobi-determinant:

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2 \sin\phi$$

$$\iiint_A f(x,y,z)\,dx\,dy\,dz$$
$$= \iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

Et kuleskall er gitt ved

$$\mathbf{r}(\theta,\phi) = \cos\theta\sin\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\phi\,\mathbf{k}$$

hvor $0 \le \theta \le 2\pi$ og $0 \le \phi \le \pi$.

$$\begin{aligned}
\mathbf{r}_\theta &= -\sin\theta\sin\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} \\
\mathbf{r}_\phi &= \cos\theta\cos\phi\,\mathbf{i} + \sin\theta\cos\phi\,\mathbf{j} - \sin\phi\,\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}_\theta \times \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \end{vmatrix} \\
&= \cos\theta\sin^2\phi\,\mathbf{i} + \sin\theta\sin^2\phi\,\mathbf{j} + \cos\phi\sin\phi\,\mathbf{k}
\end{aligned}$$

Betrakt vektorfeltet $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Vi har

$$
\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_S \mathbf{F}(\mathbf{r}(\theta,\phi))\cdot \mathbf{r}_\theta \times \mathbf{r}_\phi \, d\theta\, d\phi \\
&= \int_0^{2\pi}\int_0^{\pi} (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi) \\
&\qquad \cdot (\cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \cos\phi\sin\phi)\, d\theta\, d\phi \\
&= \int_0^{2\pi}\int_0^{\pi} (\sin^3\phi + \cos^2\phi\sin\phi)\, d\theta\, d\phi \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\theta\, d\phi \\
&= 2\pi\int_0^{\pi} \sin\phi\, d\phi \\
&= 2\pi[-\cos\theta]_0^{\pi} = 4\pi
\end{aligned}
$$

Vektorfelt

$$\mathbf{F}(x,y,z) = 2\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

over flaten

$$\mathbf{r}(u,v) = (u+v)\mathbf{i} + (u-v)\mathbf{j} + uv\mathbf{k}, \quad 0 \le u,v \le 1$$

$$\mathbf{r}_u = \mathbf{i} + \mathbf{j} + v\mathbf{k}, \quad \mathbf{r}_u = \mathbf{i} - \mathbf{j} + u\mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_u = (u+v)\mathbf{i} - (u-v)\mathbf{j} - 2\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(u,v)) = \mathbf{F}(u+v, u-v, uv) = 2\mathbf{i} + (u+v)\mathbf{j} + (u-v)uv\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2u + 2v - u^2 + v^2 - 2u^2v + 2uv^2$$

$$\iint_S \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dS = \int_0^1 \int_0^1 2u + 2v - u^2 + v^2 - 2u^2 v + 2uv^2 \, du\, dv$$
$$= 2$$

Divergensen til $\mathbf{F} = P\,\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ er definert ved

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

For eksemplet fra forrige side:

$$\nabla \cdot \mathbf{F} = \nabla \cdot (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\iiint_{S^3} \nabla \cdot \mathbf{F}\, dV = 3 \cdot \frac{4 \cdot 1^3}{3} = 4\pi$$

Det er ikke tilfeldig at disse er like.

En terning $Q : 0 \leq x, y, z \leq 1$ og et vektorfelt $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + z\,\mathbf{j} + z^2\,\mathbf{k}$.
Terningen har 6 flater:

1. $f_1 : 0 \leq x, y \leq 1,\ z = 0,\ \mathbf{n}_1 = -\mathbf{k},\ \mathbf{F}(x,y,0) = xy\,\mathbf{i}$

$$\iint_{f_1} \mathbf{F} \cdot \mathbf{n}\, dS = 0$$

2. $f_2 : 0 \leq x, y \leq 1,\ z = 1,\ \mathbf{n}_2 = \mathbf{k},\ \mathbf{F}(x,y,1) = xy\,\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\iint_{f_2} \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 dS = 1$$

3. $f_3 : 0 \leq x, z \leq 1,\ y = 0,\ \mathbf{n}_3 = -\mathbf{j},\ \mathbf{F}(x,y,1) = z\,\mathbf{j} + z^2\,\mathbf{k}$

$$\iint_{f_3} \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 (-z)\, dx\, dz = -\frac{1}{2}$$

4. $f_4: 0 \le x, z \le 1,\ y = 1,\ \mathbf{n}_4 = \mathbf{j},\ \mathbf{F}(x,1,z) = x\,\mathbf{i} + z\,\mathbf{j} + z^2\,\mathbf{k}$

$$\iint_{f_4} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1 \int_0^1 z\,dx\,dz = \frac{1}{2}$$

5. $f_5: 0 \le y, z \le 1,\ x = 0,\ \mathbf{n}_5 = -\mathbf{i},\ \mathbf{F}(0,y,z) = z\,\mathbf{j} + z^2\,\mathbf{k}$

$$\iint_{f_5} \mathbf{F}\cdot\mathbf{n}\,dS = 0$$

6. $f_6: 0 \le y, z \le 1,\ x = 1,\ \mathbf{n}_6 = \mathbf{i},\ \mathbf{F}(1,y,z) = y\,\mathbf{i} + z\,\mathbf{j} + z^2\,\mathbf{k}$

$$\iint_{f_6} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1 \int_0^1 y\,dy\,dz = \frac{1}{2}$$

som gir

$$\iint_{Q} \mathbf{F}\cdot\mathbf{n}\,dS = \sum_{j=1}^{6} \iint_{f_j} \mathbf{F}\cdot\mathbf{n}\,dS = 0 + 1 - \frac{1}{2} + \frac{1}{2} + 0 + \frac{1}{2} = \frac{3}{2}$$

Divergensen til vektorfeltet er

$$\nabla \cdot \mathbf{F} = y + 0 + 2z$$

og

$$\begin{aligned}
\int_0^1 \int_0^1 \int_0^1 (y + 2z)\, dx\, dy\, dz &= \int_0^1 \int_0^1 (y + 2z)\, dy\, dz \\
&= \int_0^1 [yz + z^2]_0^1\, dy \\
&= \int_0^1 (y + 1)\, dy \\
&= [\frac{1}{2}y^2 + y]_0^1 = \frac{3}{2}
\end{aligned}$$

En terning $Q_\varepsilon : 0 \leq x, y, z \leq \varepsilon$ og et vektorfelt $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i}$.
Terningen har 6 flater:

1. $f_1 : 0 \leq x, y \leq \varepsilon,\ z = 0,\ \mathbf{n}_1 = -\mathbf{k},\ \mathbf{F} \cdot \mathbf{n} = 0$
2. $f_2 : 0 \leq x, y \leq \varepsilon,\ z = \varepsilon,\ \mathbf{n}_2 = \mathbf{k},\ \mathbf{F} \cdot \mathbf{n} = 0$
3. $f_3 : 0 \leq x, z \leq \varepsilon,\ y = 0,\ \mathbf{n}_3 = -\mathbf{j},\ \mathbf{F} \cdot \mathbf{n} = 0$
4. $f_4 : 0 \leq x, z \leq \varepsilon,\ y = \varepsilon,\ \mathbf{n}_4 = \mathbf{j},\ \mathbf{F} \cdot \mathbf{n} = 0$

5. $f_5 : 0 \leq y, z \leq \varepsilon,\ x = 0,\ \mathbf{n}_5 = -\mathbf{i},\ \mathbf{F}(0, y, z) = P(0, y, z)\,\mathbf{i}$

$$\iint_{f_5} \mathbf{F} \cdot \mathbf{n}\, dS = -\int_0^{\varepsilon} \int_0^{\varepsilon} P(0, y, z)\, dy\, dz$$

6. $f_6 : 0 \leq y, z \leq \varepsilon,\ x = \varepsilon,\ \mathbf{n}_6 = \mathbf{i},\ \mathbf{F}(\varepsilon, y, z) = P(\varepsilon, y, z)\,\mathbf{i}$

$$\iint_{f_6} \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{\varepsilon} \int_0^{\varepsilon} P(\varepsilon, y, z)\, dy\, dz$$

som gir

$$\begin{aligned}
\iint_{Q} \mathbf{F} \cdot \mathbf{n}\, dS &= \int_0^{\varepsilon} \int_0^{\varepsilon} \left(P(\varepsilon, y, z) - P(0, y, z) \right) dy\, dz \\
&\approx \int_0^{\varepsilon} \int_0^{\varepsilon} \varepsilon \frac{\partial P}{\partial x}(0, y, z)\, dy\, dz \\
&= \int_0^{\varepsilon} \int_0^{\varepsilon} \int_0^{\varepsilon} \frac{\partial P}{\partial x}(0, y, z)\, dx\, dy\, dz \\
&= \iiint_{Q_\varepsilon} \frac{\partial P}{\partial x}(0, y, z)\, dV = \iiint_{Q_\varepsilon} \nabla \cdot \mathbf{F}\, dV
\end{aligned}$$

Vi kan gjøre det samme for vektorfelt i y-retning og z-retning, og vi kan dekke et område i (x, y, z)-rommet med sånne små Q_ε-terninger. Det gir oss **divergensteoremet**:

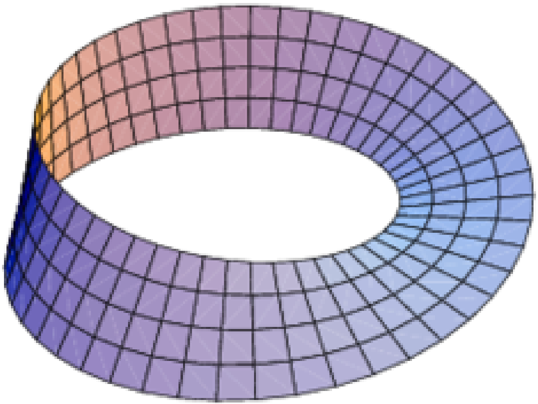
Setning

La V være et lukket, begrenset stykkvis glatt område i rommet med utadrettet enhetsnormalvektor $\mathbf{n}$. Anta at $\mathbf{F}$ er et vektorfelt med kontinuerlige partieltderiverte i et åpent område som inneholder V. Da har vi

$$\iiint_V \nabla \cdot \mathbf{F} \, dx \, dy \, dz = \int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dV$$

Vektorfelt langs en linje (x-aksen) $\mathbf{F}(x) = f(x)\mathbf{i}$. Betrakter intervallet $[A, B]$. Normalvektoren svarer til vektoren $\mathbf{i}$ i $x = B$ og $-\mathbf{i}$ i $x = A$. Det gir fluks $\mathbf{F} \cdot \mathbf{n}(A) = -f(A)$ og $\mathbf{F} \cdot \mathbf{n}(B) = f(B)$. Samlet fluks blir $f(B) - f(A)$. Divergensen $\nabla \cdot \mathbf{F} = f'(x)$ og divergensteoremet gir

$$\int_A^B f'(x)\,dx = f(B) - f(A)$$



Möbiusbånd, et eksempel på en **ikke-orienterbar flate**.