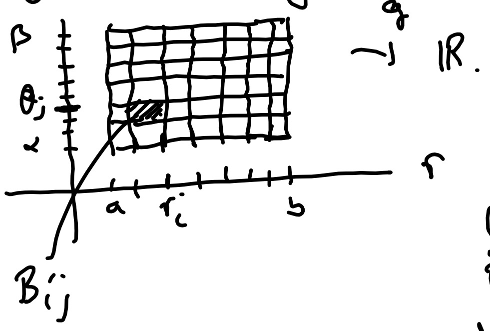
6.3. Polarkoordinater

La $R = [a,b] \times [\alpha, \beta]$ være en kube i (r,θ)-planet. La $g: R \to \mathbb{R}$ være kontinuerlig.



Partisjoner $[a,b]$ og $[\alpha,\beta]$

$a = r_0 < r_1 < r_2 < \dots < r_n = b$

$\alpha = \theta_0 < \theta_1 < \dots < \theta_n = \beta$

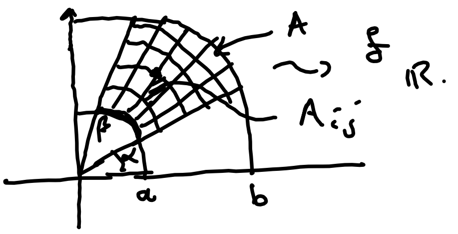
i like store biter.

$$\iint_R g(x,y)\,dx\,dy = \lim_{n\to\infty} \sum_{1\le i,j\le n} g(r_i,\theta_j)\cdot|B_{ij}|,$$

der $|B_{ij}|$ er arealet til B_{ij}.

Anta nå at vi ønsker å integrere en kontinuerlig $f: A \to \mathbb{R}$, der A er definert i polarkoordinater

$$a \le r \le b, \quad \alpha \le \theta \le \beta.$$



$$\iint_A f(x,y)\,dx\,dy = \lim_{n\to\infty} \sum_{1\le i,j\le n} f(r_i\cos\theta_j, r_i\sin\theta_j)\cdot|A_{ij}|$$

Hva er $|A_{ij}|$?

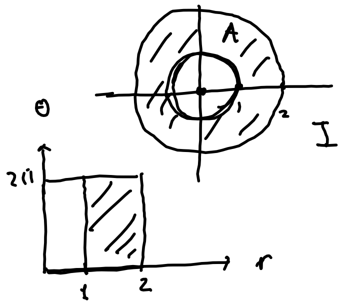
$$\approx \lim_{n\to\infty} \sum_{1\le i,j\le n} \underbrace{f(r_i\cos\theta_j, r_i\sin\theta_j)\cdot r_i}_{g}\cdot|R_{ij}|$$

$$= \iint_R f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta.$$

SETNING 6.3.1

$$\iint_A f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta.$$

EKSEMPEL 1 La $A = \{(x,y) : 1 \le x^2 + y^2 \le 2\}$, og la $f(x,y) = \frac{1}{x^2+y^2}$.



Finn $I = \iint_A f(x,y)\,dx\,dy$.

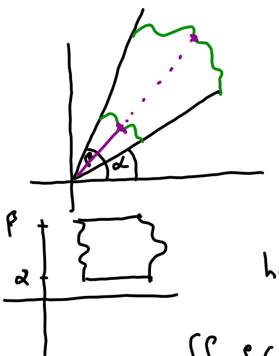
$$I = \int_0^{2\pi}\int_1^2 f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \frac{1}{r^2\cos^2\theta + r^2\sin^2\theta}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \frac{1}{r^{\cancel{2}}}\cdot \cancel{r}\,dr\,d\theta$$

$$= 2\pi\,\Big[\log r\Big]_1^2 = 2\pi(\log 2 - \log 1)$$

$$= 2\pi \log 2.$$



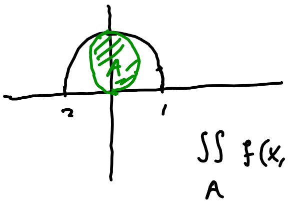
La $\phi_1, \phi_2 : [\alpha, \beta] \to \mathbb{R}_+$ være kontinuerlige funksjoner med $\phi_1 \le \phi_2$.

Sett
$$A = \{(r\cos\theta, r\sin\theta) : \alpha \le \theta \le \beta,\ \phi_1(\theta) \le r \le \phi_2(\theta)\}$$

Dersom $f : A \to \mathbb{R}$ er kontinuerlig har vi at

$$\iint_A f(x,y)\,dx\,dy = \int_\alpha^\beta \left(\int_{\phi_1(\theta)}^{\phi_2(\theta)} f(r\cos\theta, r\sin\theta)\cdot r\,dr\right) d\theta$$

EKSEMPEL 2: La A være området i planet avgrenset av kurven
$$(\sin\theta\cdot\cos\theta,\ \sin\theta\sin\theta)$$
der $0 \le \theta \le \pi$.



La $f(x,y) = \sqrt{x^2+y^2}$.

$$\iint_A f(x,y)\,dx\,dy = \int_0^\pi \left(\int_0^{\sin\theta} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\cdot r\,dr\right) d\theta$$

$$= \int_0^\pi \left(\int_0^{\sin\theta} r^2\,dr\right) d\theta$$

$$= \int_0^\pi \Big[\tfrac{1}{3} r^3\Big]_0^{\sin\theta}\,d\theta$$

$$= \int_0^\pi \tfrac{1}{3}\sin^3\theta\,d\theta$$

$$= 4/\ldots$$

6.4. Noen anvendelser av integralet

6.4.1 Arealberegninger.

Samme som forrige eksempel, altså A er mengden avgrenset av kurven
$$(\sin\theta\cos\theta, \sin\theta\sin\theta)$$
der $0 \le \theta \le \pi$.

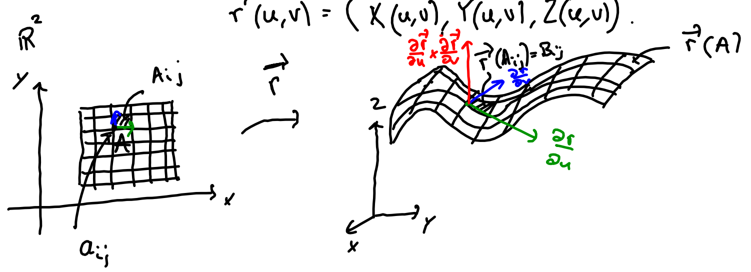
Vi integrerer funksjonen f som er konstant lik 1 over A:

$$\text{Arealet} = \iint_A 1 \, dx dy$$
$$\overset{\text{Polarkoord.}}{=} \int_0^\pi \left(\int_0^{\sin\theta} 1 \cdot r \, dr \right) d\theta$$
$$= \int_0^\pi \left[\tfrac{1}{2} \cdot r^2 \right]_0^{\sin\theta} d\theta$$
$$= \tfrac{1}{2} \int_0^\pi \sin^2\theta \, d\theta = \pi/4 .$$

$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$

6.4.3 Arealberegninger av flater i $\mathbb{R}^3$.

En parametrisert flat i $\mathbb{R}^3$ består av et område $A \subset \mathbb{R}^2$ og en kontinuerlig funksjon
$$\vec{r}(u,v) = (X(u,v), Y(u,v), Z(u,v)).$$



$$\text{Arealet}(\vec{r}(A)) = \sum_{ij} \underbrace{|B_{ij}|}_{\text{arealet til } B_{ij}}.$$

Hva er ca. arealet til B_{ij} når partisjonen er veldig fin.

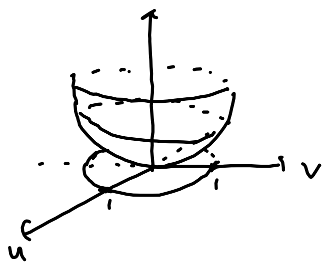
Svar: $|B_{ij}| \approx |\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}| \cdot |A_{ij}|$

Da burde arealet til $\vec{r}(A)$ være tilnærmet
$$\sum_{ij} \cdot \left|\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial r}{\partial v}\right)(a_{ij})\right| \cdot |A_{ij}|$$

Definisjon: Arealet til flaten er gitt ved
$$\iint_A \left|\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial r}{\partial v}\right)(u,v)\right| \, du dv .$$

EKSEMPEL 4

Finn arealet av paraboloiden $f(u,v) = u^2 + v^2$ som ligger over enhetsdisken i $\mathbb{R}^2$.



Her er

$$A = \{(u,v) : u^2 + v^2 \leq 1\}.$$

Vi kan bruke parametriseringen

$$\vec{r}(u,v) = (u, v, f(u,v)) = (u, v, u^2+v^2).$$

$$\frac{\partial \vec{r}}{\partial u}(u,v) = (1, 0, 2u)$$

$$\frac{\partial \vec{r}}{\partial v}(u,v) = (0, 1, 2v)$$

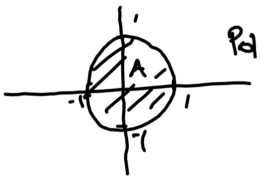
$$\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix}$$

$$= i \cdot (-2u) - j \cdot 2v + k \cdot 1$$

$$= (-2u, -2v, 1)$$

$$\left\| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right\| = \sqrt{4u^2 + 4v^2 + 1^2} = \sqrt{1 + 4u^2 + 4v^2}$$

$$\text{Arealet} = \iint_A \sqrt{1 + 4u^2 + 4v^2}\, du\, dv.$$



Polarkoord

$$= \int_0^{2\pi} \int_0^1 (1+4r^2)^{1/2} \cdot r\, dr\, d\theta$$

$$= 2\pi \int_0^1 (1+4r^2)^{1/2} r\, dr$$

$$= 2\pi \left[(1+4r^2)^{3/2} (1/12) \right]_0^1$$

$$= (\pi/6)(5\sqrt{5} - 1).$$