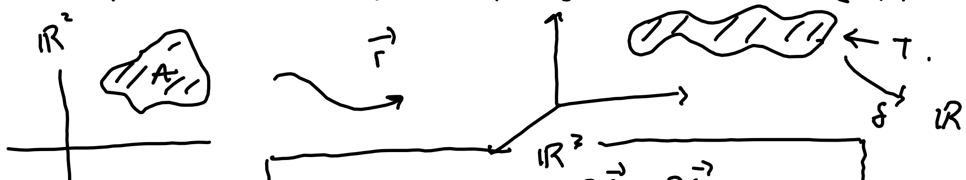
FLATEINTEGRALER AV SKALARFELT

Husk at dersom $\vec{r}: A \to \mathbb{R}^3$ er en parametrisert flate, og skriv $T = \vec{r}(A)$.



$$\boxed{\text{Areal}(T) = \iint_A \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| \, du\,dv}$$

DEF 6.4.1 La f være kontinuerlig på T. Da definerer vi integralet til f over T til å være

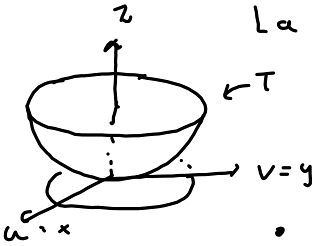
$$\iint_T f \cdot dS = \iint_A f(\vec{r}(u,v)) \cdot \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}(u,v)\right| \, du\,dv.$$

EKS: La $A = \{(u,v) : u^2 + v^2 < 1\}$.

La $\vec{r}(u,v) = (u, v, u^2+v^2)$.

$f(x,y,z) = x y z^2$.

Regn ut $\iint_T f \, dS$,



- $\frac{\partial \vec{r}}{\partial u}(u,v) = (1, 0, 2u)$

  $\frac{\partial \vec{r}}{\partial v}(u,v) = (0, 1, 2v)$

$$\left|\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v)\right| = \sqrt{1+4u^2+4v^2}$$

- $f(\vec{r}(u,v)) = uv(u^2+v^2)^2$

$$\iint_T f\,dS = \iint_A \sqrt{1+4u^2+4v^2} \cdot u \cdot v \cdot (u^2+v^2)^2 \, du\,dv$$

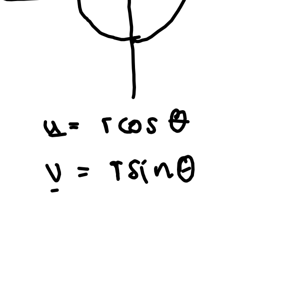
Polarkoord

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\cos\theta\, r\sin\theta \cdot r^4 \cdot r \, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2} \cdot r^7 \cdot \cos\theta \sin\theta \, dr\,d\theta$$

$$= \int_0^1 \sqrt{1+4r^2}\, r^7 \underbrace{\left(\int_0^{2\pi} \cos\theta \sin\theta \, d\theta\right)}_{=0} dr$$

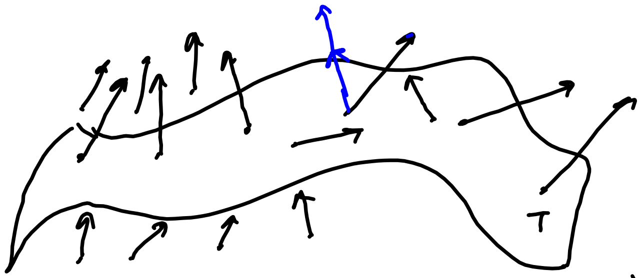
$$= 0$$



$u = r\cos\theta$

$v = r\sin\theta$

INTEGRAL AV VEKTORFELTER OVER FLATER.



← Parametrisert flate.

Vektorfelt $\vec{F}$.

Vi tenker oss $\vec{F}$ som hastighetsfeltet til en flytende væske.

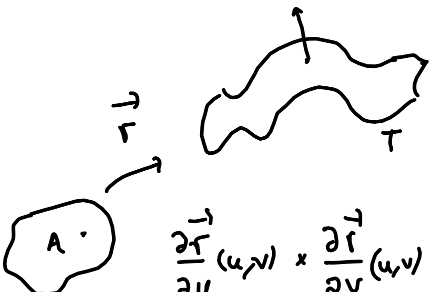
Hvordan kan vi måle hvor mye væske som passerer gjennom T?

Vi "burde" integrere lengden til normalkomponenten til vektorfeltet over T:

Velg et kontinuerlig vektorfelt med lengde 1, $\vec{n}$ på T som står normalt på T. Definer integralet av $\vec{F}$ over T som

(*) $$\boxed{\iint_T \vec{F}\cdot\vec{n}\, dS}$$

Hvordan regne det ut?

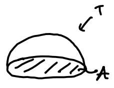


$$\vec{n}(u,v) = \frac{\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v)}{\left|\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial u}(u,v)\right|}$$

$$(*) = \iint_A \vec{F}(\vec{r}(u,v)) \cdot \frac{\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)(u,v)}{\left|\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)(u,v)\right|} \cdot \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}(u,v)\right| du\, dv$$

$$= \iint_A \vec{F}(\vec{r}(u,v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)(u,v)\, du\, dv.$$

EKSEMPEL: La T være den nordlige halvkule i $\mathbb{R}^3$.



La $\vec{F}(x,y,z) = (x^2 z, y^2 z, z^2)$

Vi parametriserer T over $\{(u,v) : u^2+v^2<1\} = A$,

via $\vec{r}(u,v) = (u, v, (1-u^2-v^2)^{1/2})$

- $\frac{\partial \vec{r}}{\partial u}(u,v) = (1, 0, -u(1-u^2-v^2)^{-1/2})$
- $\frac{\partial \vec{r}}{\partial v}(u,v) = (0, 1, -v(1-u^2-v^2)^{-1/2})$
- $\frac{\partial \vec{r}}{\partial v}(u,v) = (u(1-u^2-v^2)^{-1/2}, v(1-u^2-v^2)^{-1/2}, 1)$

Vi får: $\iint_T \vec{F}\cdot\vec{n}\, dS = \iint_A F(\vec{r}(u,v))\cdot(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v})(u,v)\, du\,dv$

$= \iint_A (u^2(1-u^2-v^2)^{1/2}, v^2(1-u^2-v^2)^{1/2}, (1-u^2-v^2))\cdot$
$(u(1-u^2-v^2)^{-1/2}, v(1-u^2-v^2)^{-1/2}, 1)\, du\,dv$

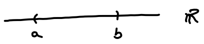
$= \iint_A u^3+v^3+1-u^2-v^2\, du\,dv$

$= \pi/3$.

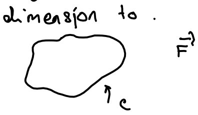
6.5. Greens teorem.

Husk fra Kalkulus: dersom $f:[a,b]\to\mathbb{R}$ er deriverbar har vi at

$$\int_a^b f'(x)\,dx = f(b)-f(a).$$



Greens teorem er en "generalisering" til dimensjon to.



La $\vec{F} = (P,Q)$ være et vektorfelt nær en kurve C med stykkevis glatt parametrisering $\vec{r}(t) = (x(t), y(t))$.

$\int_C \vec{F}\cdot d\vec{r} = \int \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

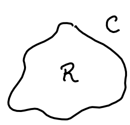
$= \int (P(\vec{r}(t)), Q(\vec{r}(t)))\cdot(x'(t), y'(t))\,dt$

$= \int P(\vec{r}(t))\cdot x'(t) + Q(\vec{r}(t))\cdot y'(t)\,dt$.

Det er vanlig å skrive

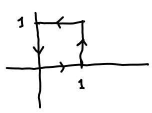
$$\boxed{\int_C \vec{F}\,d\vec{r} = \int_C P\,dx + Q\,dy}$$

Teorem 6.5.1 (Greens teorem).
Anta at C er en enkel lukket kurve med en stykkevis glatt parametrisering $\vec{r}$. La R være området avgrenset av C. La $\vec{F} = (P,Q)$ være et vektorfelt nær R der de partiellderiverte til P og Q er kontinuerlige.



Da:
$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$
når C er parametrisert mot klokka

EKS: La $\vec{F}(x,y) = (xy^2, x^2y)$, og integrer $\vec{F}$ over randa til $[0,1]^2$.

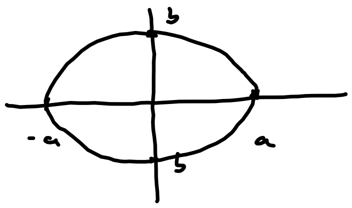


$P(x,y) = xy^2$ $\quad \frac{\partial P}{\partial y} = 2xy$

$Q(x,y) = x^2y$ $\quad \frac{\partial Q}{\partial x} = 2xy$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ så $\int \vec{F}\cdot\vec{r}$ ...

<u>EKS</u> Finn arealet til området A avgrenset av ellipsen C, $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.



Anta at vi har et vektorfelt $\vec{F} = (P, Q)$ s.a. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \underline{1}$.

$$\int_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_A dx\,dy$$

$= \text{arealet}$.

Set $Q(x,y) = x/2$

$P(x,y) = -y/2$

$$\int_C P\,dx + Q\,dy = \ldots = a\,b\cdot\pi.$$

$\vec{r}(t) = (a\cos t, b\sin t)$

$t \in [0, 2\pi]$