LINEARISERING OG KJERNEREGELEN 2.7 & 2.8.

DEF 2.8.2. La $A \subseteq \mathbb{R}^n$, $\vec{a} \in A$, $F: A \to \mathbb{R}^n$ deriverbar i $\vec{a}$.
Affinavbildingen

$$\boxed{T_{\vec{a}}F(x) = F(\vec{a}) + F'(\vec{a}) \cdot (\vec{x} - \vec{a})}$$

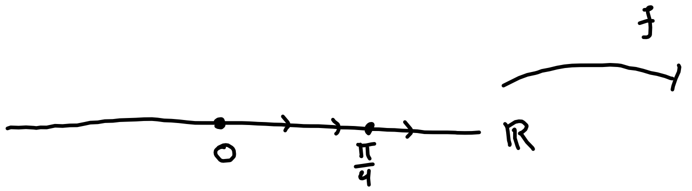
kalles lineariseringen til F i $\vec{a}$.

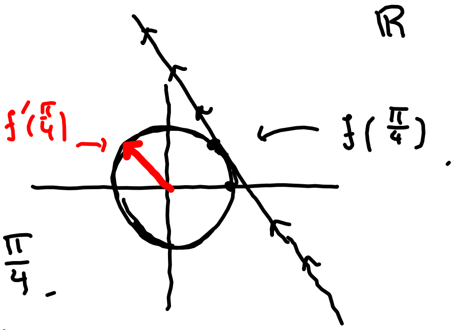
Vi har:

$$\frac{\sigma(x)}{\|\vec{x} - \vec{a}\|} := \frac{F(\vec{x}) - T_{\vec{a}}F(x)}{\|\vec{x} - \vec{a}\|} \xrightarrow[\vec{x} \to \vec{a}]{} 0.$$

Så $F(x) = T_{\vec{a}}F(\vec{x}) + \sigma(x)$.

EKSEMPEL: La $f(x) = (\cos(x), \sin(x))$,
$f: \mathbb{R} \to \mathbb{R}^2$.





Finn lineariseringen til f i $\frac{\pi}{4}$.

Konstantleddet: $f(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

$$f'(\tfrac{\pi}{4}) = \begin{bmatrix} -\sin(\frac{\pi}{4}) \\ \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\boxed{T_{\frac{\pi}{4}}f(x) = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} + \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \cdot [x - \tfrac{\pi}{4}].}$$

EKSEMPEL 2 Vi ser på avbildningen $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ gitt ved

$$F(x,y) = \left(3/2 - 3/2\,x + x^2,\ 7/4 - 3/4\,y - x + xy\right)$$

Hvordan "oppfører" F seg i nærheten av (1,1)?

Hva skjer med iteratene $F^n = \underbrace{F \circ F \circ \dots \circ F}_{n \text{ ganger}}$

når $n \longrightarrow \infty$?

Vi ser på lineariseringen til F.

Konstantleddet: $F(1,1) = (1,1)$.

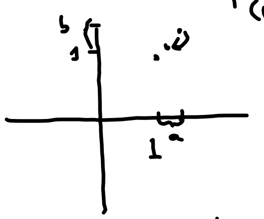
$$F'(x,y) = \begin{bmatrix} -3/2 + 2x & 0 \\ -1 + y & -3/4 + x \end{bmatrix}$$

$$F'(1,1) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

$$\boxed{T_{(1,1)}F(x,y) = (1,1) + \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}}$$

Skriv om litt:
Skriv $x = 1+a$, $y = 1+b$.

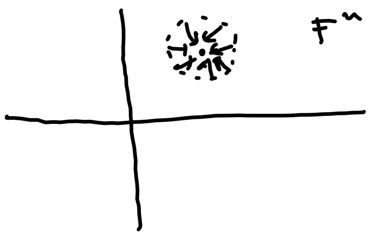
$$T_{(1,1)}F(1+a, 1+b) = (1,1) + \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$



$$= (1,1) + (a/2,\ b/4)$$

$$= (1 + a/2,\ 1 + b/4)$$

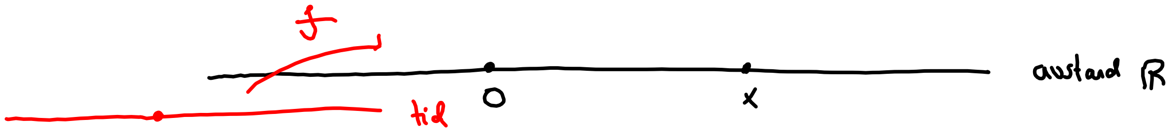
Så alle punkter trekkes mot (1,1).



F^n $n \to \infty$

2.7. Kjerneregelen.

Tenk deg at en person beveger seg langs en smal sti.



La $g(x)$ betegne temperaturen i $x \in \mathbb{R}$.

La $f(t)$ betegne avstanden fra utgangspunktet 0.

Hvordan forandrer temperaturen seg for personen ved tid t?

Temperatur: $h(t) = g(f(t))$.

Hva er $h'(t)$?

Eksempel: $g(x) = 10 + 2x$ (temp. stiger med 2° per meter)

$f(t) = t$ (går 1 meter per sekund).

$h'(t) = 2$.

Hva hvis $f(t) = 2t$?

$h'(t) = 4$.

I begge tilfeller: $h'(t) = g'(f(t)) \cdot f'(t)$.

Hva med generelle funksjoner f og g?

Derivert til $h = g \circ f$ i $a \in \mathbb{R}$.

Vi har

$$f(x) = f(a) + f'(a)(x-a) + o_f\cancel{(x)}$$

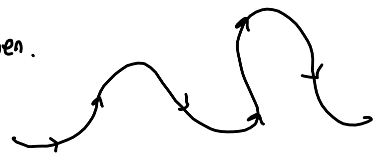
$$g(y) = g(f(a)) + g'(f(a))(y - f(a)) + o_g\cancel{(y)}.$$

Burde få at $h'(a) = g'(f(a)) \cdot f'(a)$.

Hva hvis vi ser på 2 dimensjoner?

$f(t) = (x(t), y(t))$

$g(x,y)$ er temperaturen.



Må se på

$$h(t) = g(x(t), y(t)) = (g \circ f)(t).$$

Det viser seg at vi har en tilsvarende formel:

$h'(t)$?

$$h'(t) = g'(f(t)) \cdot f'(t)$$

- $g'(x,y) = \begin{bmatrix} \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{bmatrix}$

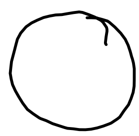
- $f'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$

$$\boxed{\begin{aligned} h'(t) &= \begin{bmatrix} \frac{\partial g}{\partial x}(f(t)) & \frac{\partial g}{\partial y}(f(t)) \end{bmatrix} \cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \\ &= \frac{\partial g}{\partial x}(f(t)) \cdot x'(t) + \frac{\partial g}{\partial y}(f(t)) \cdot y'(t) \\ &= \nabla g(f(t)) \cdot f'(t). \end{aligned}}$$

<u>Eksempel</u> $g(x,y) = x^4 + 2x^2y^2 + y^4$

$f(t) = (\cos(t), \sin(t))$

Se på $h(t) = g(f(t))$.



$$\frac{\partial g}{\partial x}(x,y) = 4x^3 + 4xy^2$$

$$\frac{\partial g}{\partial y}(x,y) = 4y^3 + 4x^2y$$

$$\begin{aligned} h'(t) &= (4\cos^3 t + 4\cos t \sin^2 t) \cdot (-\sin t) \\ &\quad + (4\sin^3 t + 4\cos^2 t \sin t) \cdot (\cos t) \\ &= -4\cos^3 t \cdot \sin t - 4\cos t \sin^3 t \\ &\quad + 4\sin^3 t \cos t + 4\cos^3 t \cdot \sin t \\ &= 0. \end{aligned}$$

<u>Merk:</u> $g(x,y) = x^4 + 2x^2y^2 + y^4 = (x^2+y^2)^2$,

så $g(f(t)) \equiv 1$.

$$\mathbb{R} \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R} \qquad h = g \circ f .$$

$$h'(x) = \frac{\partial g}{\partial x_1}(f(x)) \cdot \frac{\partial f_1}{\partial x}(x) + \frac{\partial g}{\partial x_2}(f(x)) \cdot \frac{\partial f_2}{\partial x}(x) \ldots + \frac{\partial g}{\partial x_n}(f(x)) \cdot \frac{\partial f_n}{\partial x}(x)$$

$$= \sum_{l=1}^{n} \frac{\partial g}{\partial x_l}(f(x)) \cdot \frac{\partial f_l}{\partial x}(x) = \nabla g(f(x)) \cdot f'(x) .$$

$$\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R}$$

$$\frac{\partial h}{\partial x_j}(x) = \sum_{l=1}^{n} \frac{\partial g}{\partial x_l}(f(x)) \cdot \frac{\partial f_l}{\partial x_j}(x)$$

$$\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R}^k$$

$$\frac{\partial h_i}{\partial x_j}(x) = \sum_{l=1}^{n} \frac{\partial g_i}{\partial x_l}(f(x)) \cdot \frac{\partial f_l}{\partial x_j}(x) .$$

<u>På matriseform:</u> $h'(x) = g'(f(x)) \cdot f'(x)$.