Seksjon 3.5: Gradienter og konservative felt

Anta $\phi(x_1, \dots, x_n)$ er et skalarfelt i n variable.

Vektorfeltet $\nabla \phi = \left(\frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n} \right)$ kalles gradienten til ϕ.

Hvis $\vec{F}(\vec{x}) = \nabla \phi(\vec{x})$ for en ϕ, og for alle $\vec{x}$ (i def. mengden A)

så sier vi at ϕ er en potensialfunksjon for $\vec{F}$.

Hvis $\vec{F}$ har en potensialfunksjon, så sier vi at $\vec{F}$ er konservativt

Teorem 3.5.7 Anta $\vec{F}$ har kont. part. der. i A, og at A er åpent og enkeltsammenhengende (sammenhengde, og "uten huller

Da gjelder: $\vec{F}$ konservativ $\iff \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$

Bevis: $\Rightarrow$: Hvis $\vec{F} = \nabla \phi$ så er

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial \left(\frac{\partial \phi}{\partial x_i} \right)}{\partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i}$$

$\shortparallel$ Vi vet dette.

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial \left(\frac{\partial \phi}{\partial x_j} \right)}{\partial x_i} = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \quad \blacksquare$$

$\Leftarrow$: Hopper over.

Setning 3.5.1 Anta $\phi : A \subseteq \mathbb{R}^n \to \mathbb{R}$, at $\nabla \phi$ er kontinuerlig, og $\vec{r} : [a,b] \to A$ er en stykkevis glatt parametrisering av en kurve $\mathcal{C}$. Da er

$$\int_{\mathcal{C}} \nabla \phi \cdot d\vec{r} = \phi(\vec{r}(b)) - \phi(\vec{r}(a))$$

Bevis: Vi antar for enkelhets skyld: $\vec{r}$: deriverbar over alt, og består av ett segment kun.

$$\int_{\mathcal{C}} \nabla \phi \cdot d\vec{r} \overset{\text{def}}{=} \int_a^b \nabla \phi(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\overset{\text{kjerneregel}}{=} \int_a^b \left(\phi(\vec{r}(t))\right)' \, dt$$

$$\overset{\text{analysens fund. teorem}}{=} \left[\phi(\vec{r}(t)) \right]_a^b$$

$$= \phi(\vec{r}(b)) - \phi(\vec{r}(a)) \quad \blacksquare$$

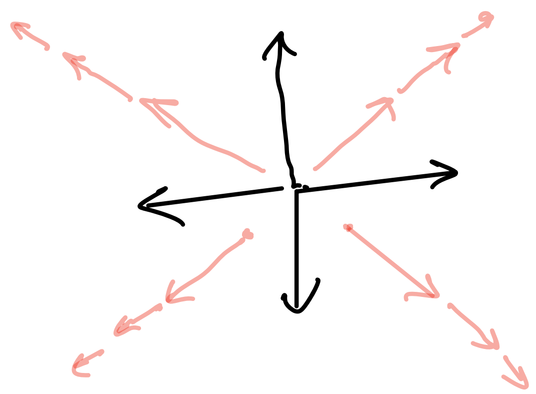
Eksempel 1: Vis at $\vec{F}(\vec{x}) = \dfrac{k\vec{x}}{|\vec{x}|^2}$ er konservativt, og har potensialfunksjon $\Phi(\vec{x}) = k \ln|\vec{x}|$.

Løsning: $\phi(\vec{x}) = k \ln\sqrt{x_1^2+\cdots+x_n^2} = k\ln(x_1^2+\cdots+x_n^2)^{\frac{1}{2}} = \frac{k}{2}\ln(x_1^2+\cdots+x_n^2)$

$$\frac{\partial\phi}{\partial x_i}(\vec{x}) = \frac{k}{\cancel{2}} \frac{\cancel{2}x_i}{x_1^2+\cdots+x_n^2} = \frac{k x_i}{x_1^2+\cdots+x_n^2} = \frac{k x_i}{|\vec{x}|^2}$$

Det følger at $\nabla\phi(\vec{x}) = \dfrac{k\vec{x}}{|\vec{x}|^2} = \vec{F}(\vec{x})$

Derfor er $\vec{F}$ konservativt.



Eksempel 2 Hvordan finne ϕ i utgangspunktet, uten å gjette?

Løsning: En potensialfunksjon må oppfylle $\nabla\phi = \vec{F}$ dvs.

$$\frac{\partial\phi}{\partial x_i} = F_i = \frac{k x_i}{|\vec{x}|^2} = \frac{k x_i}{x_1^2+\cdots+x_n^2} \Leftarrow$$

$u = x_1^2+\cdots+x_n^2$

$du = 2x_i\,dx_i$

$\frac{1}{2}du = x_i\,dx_i$

integrer m.t.p. x_i:

$$\phi(\vec{x}) = \frac{k}{2}\ln(x_1^2+\cdots+x_n^2) + \underset{\substack{\uparrow \\ \text{avh. av } x_1,\ldots,x_{i-1},\, x_{i+1},\ldots,x_n}}{C_i}$$

Hvis vi setter $C_i = 0$ får vi $\phi(\vec{x}) = \frac{k}{2}\ln(x_1^2+\cdots+x_n^2)$

Vi sjekker også at $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$:

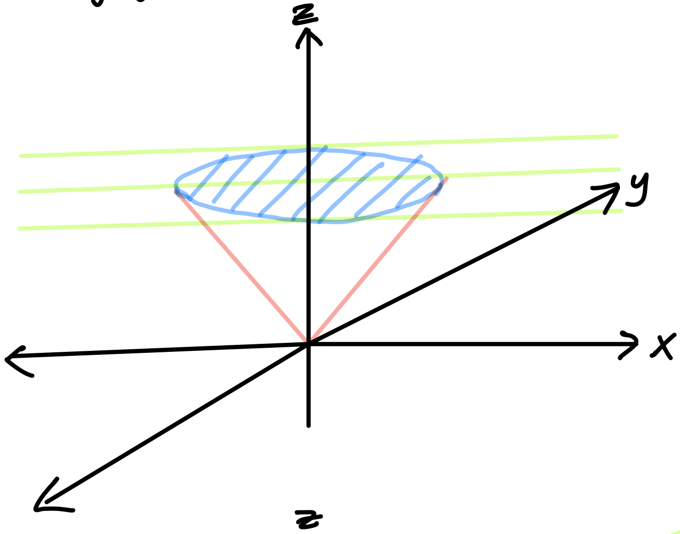
$$\frac{\partial F_i}{\partial x_j} = \frac{\partial\left(\frac{kx_i}{x_1^2+\cdots+x_n^2}\right)}{\partial x_j} = kx_i\left(-\frac{1\cdot 2x_j}{(x_1^2+\cdots+x_n^2)^2}\right) = \frac{-2kx_ix_j}{|\vec{x}|^4}$$

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial\left(\frac{kx_j}{x_1^2+\cdots+x_n^2}\right)}{\partial x_i} = kx_j\left(-\frac{2x_i}{(x_1^2+\cdots+x_n^2)^2}\right) = \frac{-2kx_ix_j}{|\vec{x}|^4},$$

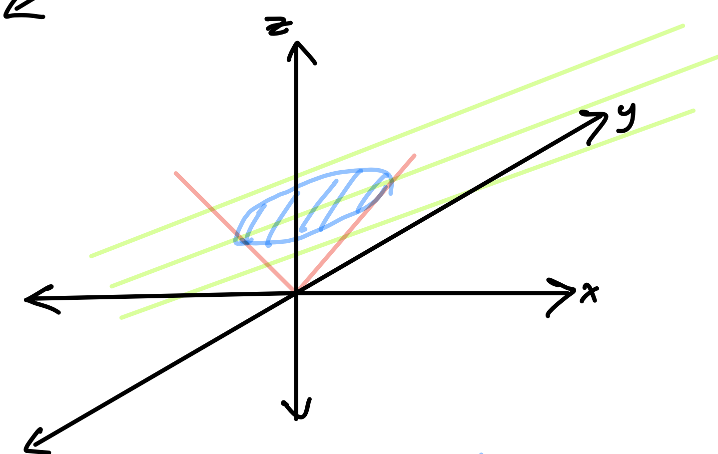
slik at disse blir like.

Seksjon 3.6

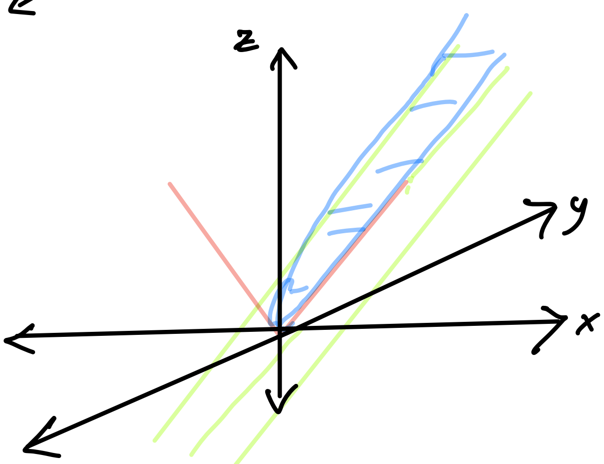
Kjeglesnitt er snittkurver mellom plan og kjegler i rommet



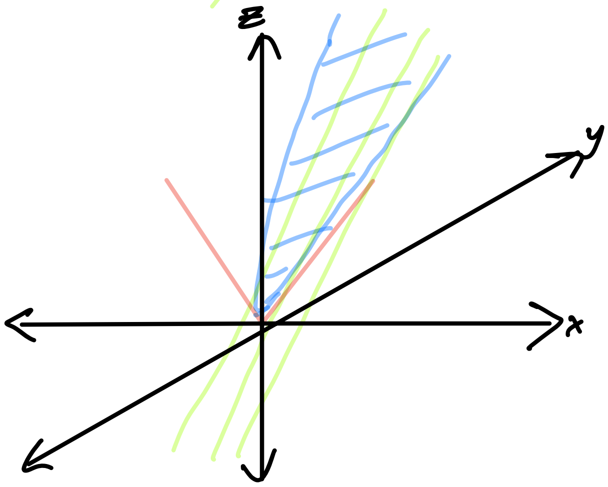
a) Flatt plan (sirkel



b) Slakt plan (ellipse



c) Plan parallelt med kjeglen (parabel)



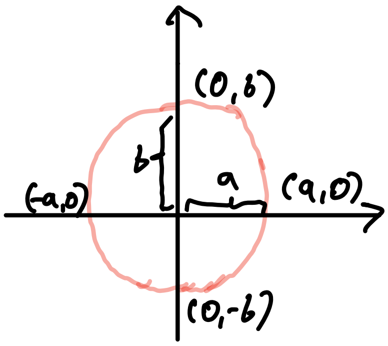
d) Enda brattere plan (hyperbel)

Standardligninger for kjeglesnitt

Ellipser:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

a og b kalles halvakser (den største kalles store halvakse.



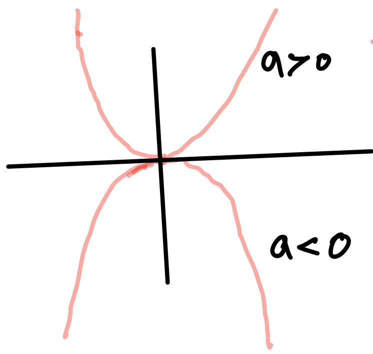
Parametrisering $\vec{r}(t) = (a\cos t, b\sin t)$
$0 \le t \le 2\pi$.

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = \right.$$

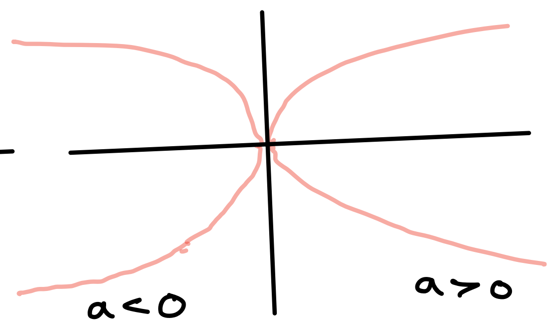
$$= \cos^2 t + \sin^2 t = 1$$

Standardlikning for parabel:

$x^2 = 4ay$



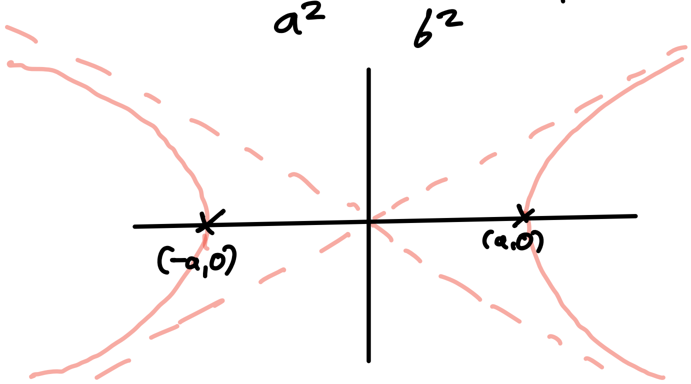
$y^2 = 4ax$



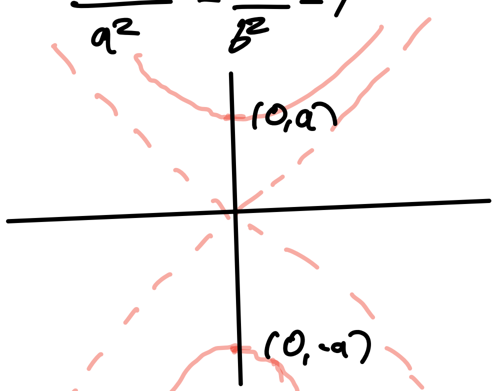
$|a|$ kalles brennvidde.

Standardlikningen for en hyperbel:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$



Parametrisering : $\vec{r}(t) = (a\cosh t \,,\, b\sinh t)$

(siden $\cosh^2 x - \sinh^2 x = 1$)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$y = \pm b\sqrt{\frac{x^2}{a^2} - 1}$$

Hvorfor har hyperbler asymptoter ?

$$\lim_{x\to\infty}\left(b\sqrt{\frac{x^2}{a^2}-1} - \frac{b}{a}x\right)$$

$$= \frac{b}{a}\lim_{x\to\infty}\left(\sqrt{x^2-a^2} - x\right)$$

$$= \frac{b}{a}\lim_{x\to\infty}\frac{\left(\sqrt{x^2-a^2}-x\right)\left(\sqrt{x^2-a^2}+x\right)}{\sqrt{x^2-a^2}+x}$$

$$= \frac{b}{a}\lim_{x\to\infty}\frac{x^2-a^2-x^2}{\underbrace{\sqrt{x^2-a^2}+x}_{\to\infty}} = \frac{b}{a}\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0.$$

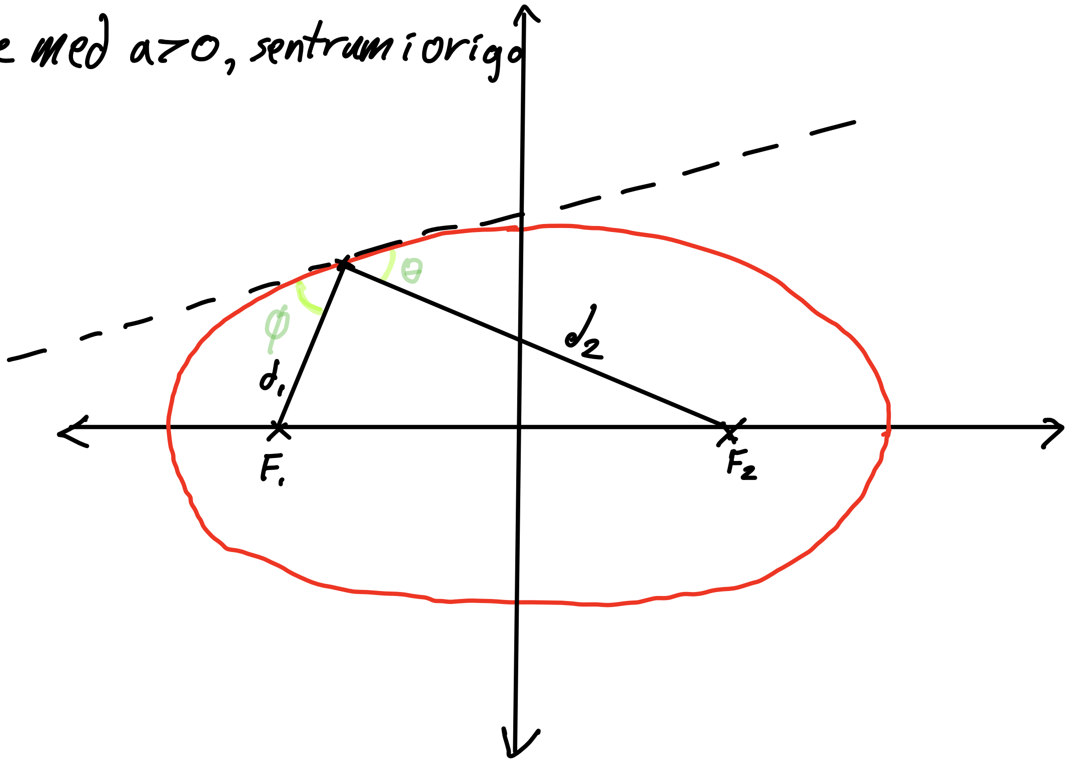
Eksempel 1

$$\frac{x^2}{\underbrace{2^2}_{a=2}} - \frac{y^2}{\underbrace{4^2}_{b=4}} = 1 \quad \left(\text{sml. med } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$$

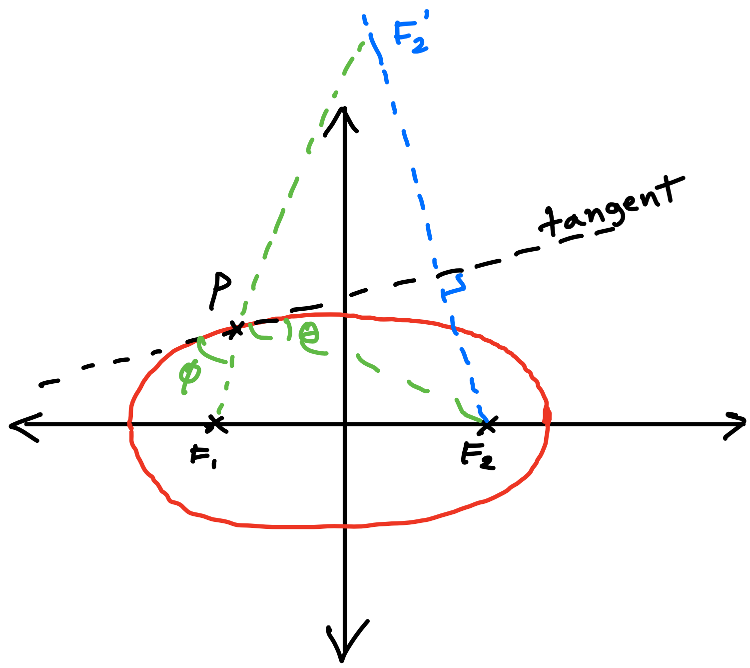
Asymptotene blir $y = \pm\frac{b}{a}x = \pm\frac{4}{2}x = \pm 2x$

Geometriske egenskaper for en ellipse I:

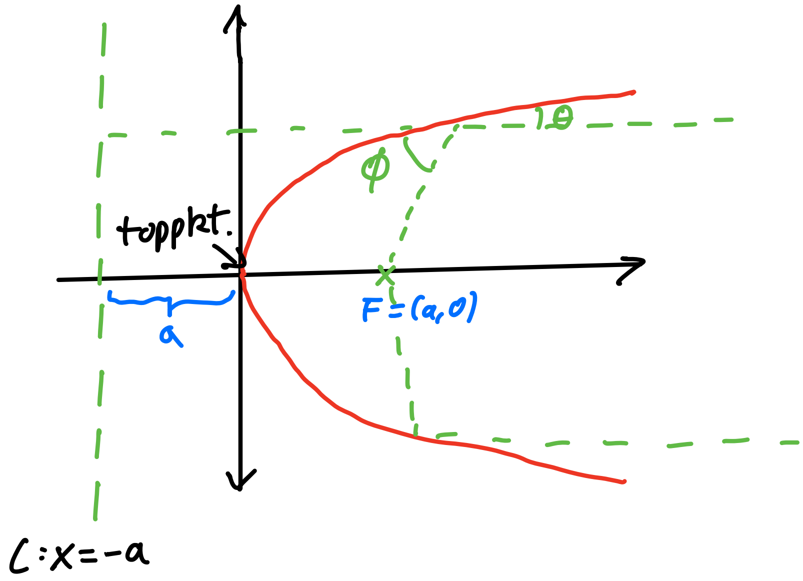
Ellipse med a>0, sentrum i origo



Bevis refleksjonsegenskapen til ellipser.



Geometriske egenskaper for en parabel



F: Brennpunkt

a: Brennvidde

Her er (0,0) toppunkt

l: Styrelinje

Geometriske egenskaper for en hyperbel