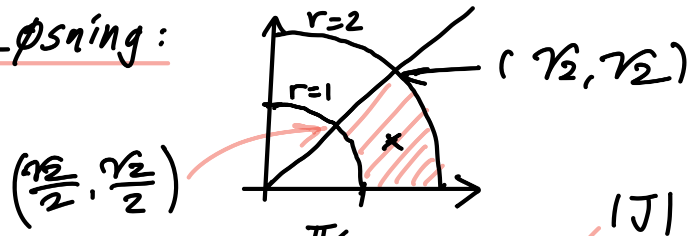
Fra sist gang:

Eksempel 4 Finn $\iint_A (x^2+y^2)\,dx\,dy$ når A er området avgrenset av $x \geq 0$, $\underbrace{1 \leq x^2+y^2 \leq 4}_{1 \leq r \leq 2}$, $\underbrace{x \geq y}_{y=x \Leftrightarrow \theta = \frac{\pi}{4}}$

Løsning:



$$\vec{T}(r,\theta) = (\overset{x}{r\cos\theta}, \overset{y}{r\sin\theta})$$

$$I = \int_1^2 \Big[\int_0^{\pi/4} (x^2+y^2)\overset{|J|}{r}\,d\theta \Big] dr$$

$$= \int_1^2 \Big[\int_0^{\pi/4} r^2 r\,d\theta \Big] dr$$

$$= \int_1^2 \Big[r^3 \theta \Big]_0^{\frac{\pi}{4}} dr$$

$$= \frac{\pi}{4} \int_1^2 r^3\,dr = \frac{\pi}{4}\Big[\frac{1}{4} r^4\Big]_1^2 = \frac{\pi}{16}(16-1) = \underline{\pi \frac{15}{16}}$$

Seksjon 6.4: Anvendelser av dobbelintegraler

1. Finne areal av et område A: $\iint_A 1\,dx\,dy$
2. Finn masse av et område A: $m(A) = \iint_A f(x,y)\,dx\,dy$, der $f(x,y)$ beskriver massetettheten

Massemiddelpunkt til A: $\bar{x} = \frac{1}{m(A)} \iint_A x f(x,y)\,dx\,dy$

$(\bar{x}, \bar{y})$

$$\bar{y} = \frac{1}{m(A)} \iint y f(x,y)\,dx\,dy$$

Med samme A som i eksempel 4: (antar massetetthet 1)

$$\text{Areal} = \iint_A 1\,dx\,dy = \int_0^{\pi/4}\left[\int_1^2 1\cdot \overset{|J|}{r}\,dr\right]d\theta = \int_0^{\pi/4}\left[\tfrac{1}{2}r^2\right]_1^2 d\theta$$

$$= \int_0^{\pi/4} \frac{3}{2}\,d\theta = \frac{\pi}{4}\cdot\frac{3}{2} = \underline{\frac{3\pi}{8}}.$$

$$\bar{x} = \overset{\frac{1}{m(A)}}{\frac{8}{3\pi}} \iint_A x\cdot 1\,dx\,dy = \frac{8}{3\pi}\int_0^{\pi/4}\left[\int_1^2 r\cos\theta\, r\,dr\right]d\theta = \frac{8}{3\pi}\int_0^{\frac{\pi}{4}}\left[\tfrac{1}{3}r^3\cos\theta\right]_1^2 d\theta$$

$$\frac{8}{3\pi}\int_0^{\frac{\pi}{4}} \frac{7}{3}\cos\theta\,d\theta = \frac{8}{3\pi}\left[\frac{7}{3}\sin\theta\right]_0^{\pi/4} = \frac{7}{3}\frac{\sqrt{2}}{2}\frac{8}{3\pi} = \underline{\frac{7}{6}\sqrt{2}}\frac{8}{3\pi}$$

$$\bar{y} = \overset{\frac{1}{m(A)}}{\frac{8}{3\pi}} \iint_A y\cdot 1\,dx\,dy = \int_0^{\frac{\pi}{4}}\left[\int_1^2 r\sin\theta\, r\,dr\right]d\theta = \frac{8}{3\pi}\int_0^{\frac{\pi}{4}}\left[\tfrac{1}{3}r^3\sin\theta\right]_1^2 d\theta$$

$$\frac{8}{3\pi}\int_0^{\pi/4} \frac{7}{3}\sin\theta\,d\theta \quad \frac{8}{3\pi}\left[-\frac{7}{3}\cos\theta\right]_0^{\frac{\pi}{4}} = -\frac{7}{3}\left(\frac{\sqrt{2}}{2}-1\right)\frac{8}{3\pi}$$

$$= \underline{\frac{7}{3}\left(1-\frac{\sqrt{2}}{2}\right)}\frac{8}{3\pi}$$

Seksjon 6.11 Anvendelser av trippelintegraler

1. Finne volum av et område i rommet: $\iiint_A 1\,dx\,dy\,dz$

2. Finne masse av et område A $\qquad m(A) = \iiint_A f(x,y,z)\,dx\,dy\,dz$,

der $f(x,y,z)$ beskriver massetetthet.

massemiddelpunkt:
$$\bar{x} = \frac{1}{m(A)}\iiint_A x\,f(x,y,z)\,dx\,dy\,dz$$
$$\bar{y} = \frac{1}{m(A)}\iiint_A y\,f(x,y,z)\,dx\,dy\,dz$$
$$\bar{z} = \frac{1}{m(A)}\iiint_A z\,f(x,y,z)\,dx\,dy\,dz$$

Prosedyre for å regne ut trippelintegraler:

1. "Tegn" området

2. Velg et koordinatsystem s.a. A kan skrives
$\{(u,v,w): a \le u \le b,\ c(u) \le v \le d(a),\ r(u,v) \le w \le s(u,v)\}$,
der c, d, r, s er kontinuerlige funksjoner
Gjør ikke dette: "Splitt opp A" i mindre biter

3. Regn ut
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$
trenger (x,y,z) uttrykt ved (u,v,w). Vi skriver $(x,y,z) = \vec{T}(u,v,w)$

4. Regn ut
$$\int_a^b \left[\int_{c(u)}^{d(u)} \left[\int_{r(u,v)}^{s(u,v)} f(\vec{T}(u,v,w))\,|J|\,dw \right] dv \right] du$$

Merk: Med $A = \{(x,y,z): a \le x \le b,\ c(x) \le y \le d(x),\ r(x,y) \le z \le s(x,y)\}$

$$\iiint_A 1\,dx\,dy\,dz = \text{volumet mellom flatene } r \text{ og } s.$$

Bevis:
$$\iiint_A 1\,dx\,dy\,dz = \int_a^b \left[\int_{c(x)}^{d(x)} \left[\int_{r(x,y)}^{s(x,y)} 1\,dz \right] dy \right] dx$$
$$= \int_a^b \left[\int_{c(x)}^{d(x)} \Big[z \Big]_{r(x,y)}^{s(x,y)} dy \right] dx$$
$$= \int_a^b \left[\int_{c(x)}^{d(x)} (s(x,y) - r(x,y))\,dy \right] dx$$

$$= \underbrace{\int_a^b \left[\int_{c(x)}^{d(x)} s(x,y)\,dy \right] dx}_{\text{volumet under flaten } s} - \underbrace{\int_a^b \left[\int_{c(x)}^{d(x)} r(x,y)\,dy \right] dx}_{\text{Volumet under flaten } r.}$$

$$\int_a^b \left[\int_{c(x)}^{d(x)} \left[\int_0^{s(x,y)} 1\,dz \right] dy \right] dx$$

Eksempel 1: Hva blir $\iiint_R 6x^3y^2z\,dx\,dy\,dz$, der $R = [0,2]\times[-1,1]\times[0,1]$?

Løsning:

$$\int_0^2 \left[\int_{-1}^1 \left[\int_0^1 6x^3y^2z\,dz \right] dy \right] dx$$

$$= \int_0^2 \left[\int_{-1}^1 \left[3x^3y^2z^2 \right]_0^1 dy \right] dx$$

$$= \int_0^2 \left[\int_{-1}^1 3x^3y^2\,dy \right] dx$$

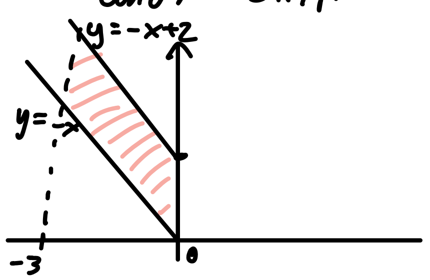
$$= \int_0^2 \left[x^3y^3 \right]_{-1}^1 dx$$

$$= \int_0^2 2x^3\,dx$$

$$= \left[\frac{1}{2}x^4 \right]_0^2 = \frac{16}{2} = \underline{8}$$

Eksempel 2 Regn ut $\iiint_A x\,dx\,dy\,dz$ når A er området avgrenset av planene $y=-x$, $y=-x+2$, $x=-3$, $x=0$, under ellipsoiden $z = x^2+2y^2$ og over $z=0$.

Løsning:



$$A = \{(x,y,z) : -3 \le x \le 0,\ -x \le y \le -x+2,\ 0 \le z \le x^2+2y^2\}$$

$$I = \int_{-3}^{0} \left[\int_{-x}^{-x+2} \left[\int_0^{x^2+2y^2} x\,dz \right] dy \right] dx$$

$$= \int_{-3}^{0} \left[\int_{-x}^{-x+2} \left[xz \right]_{0}^{x^2+2y^2} dy \right] dx$$

$$= \int_{-3}^{0} \left[\int_{-x}^{-x+2} (x^3+2xy^2)\, dy \right] dx$$

$$= \int_{-3}^{0} \left[x^3 y + \frac{2}{3} x y^3 \right]_{-x}^{-x+2} dx$$

$$= \int_{-3}^{0} \left(x^3(\cancel{-x}+2+\cancel{x}) + \frac{2}{3} x \left((-x+2)^3 + x^3 \right) \right) dx$$

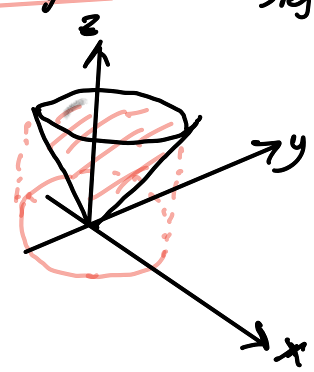
$$= \int_{-3}^{0} \left(2x^3 + \frac{2}{3} x (6x^2 - 12x + 8) \right) dx = \int_{-3}^{0} \left(6x^3 - 8x^2 + \frac{16}{3} x \right) dx$$

$$= \left[\frac{3}{2} x^4 - \frac{8}{3} x^3 + \frac{8}{3} x^2 \right]_{-3}^{0} = -\frac{243}{2} - 72 - 24 = -\frac{435}{2}$$

Eksempel 3 Hva er volumet over kjeglen $\underbrace{z = \sqrt{x^2+y^2}}_{z=r}$, og under $z=2$.

Løsning: skjæring $z=2$ og kjeglen: $\sqrt{x^2+y^2} = 2 \Leftrightarrow x^2+y^2 = 4$

$r = 2$



I sylinderkoordinater: $x = r\cos\theta$, $y = r\sin\theta$, $\underbrace{z = z}_{?}$ blir området:

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad r \le z \le 2$$

Jacobideterminanten:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$|J| = r$

$$V = \iiint_A 1\, dx\, dy\, dz = \int_0^{2\pi} \left[\int_0^{2} \left[\int_r^{2} 1 \cdot \overset{|J|}{r}\, dz \right] dr \right] d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{2} \left[rz \right]_r^2 dr \right] d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{2} (2r - r^2)\, dr \right] \partial\theta$$

$$= \int_0^{2\pi} \left[r^2 - \tfrac{1}{3} r^3 \right]_0^2 d\theta = \int_0^{2\pi} \tfrac{4}{3}\, d\theta = \underline{\tfrac{8\pi}{3}}$$

Eksempel 4 Hva er volumet til en kule med radius a?

Løsning: Vi bruker koordinater:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Kule beskrevet i polarkoordinater: $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, $0 \le \rho \le a$

$$J = \begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\rho} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\rho} \\ \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\rho} \end{vmatrix} = \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta & \sin\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta & \sin\phi\sin\theta \\ 0 & -\rho\sin\phi & \cos\phi \end{vmatrix}$$

$$= \rho\sin\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \sin\phi\cos\theta \\ \rho\sin\phi\cos\theta & \sin\phi\sin\theta \end{vmatrix} + \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix}$$

$$= \rho\sin\phi\, \rho\sin^2\phi\,(-\sin^2\theta - \cos^2\theta) + \cos\phi\, \rho^2\sin\phi\cos\phi\,(-\sin^2\theta - \cos^2\theta)$$

$$= -\rho^2\sin^3\phi - \rho^2\sin\phi\cos^2\phi$$

$$= \rho^2\sin\phi\,(-\sin^2\phi - \cos^2\phi) = -\rho^2\sin\phi \Rightarrow |J| = \rho^2\sin\phi$$

Volumet blir:

$$\int_0^{2\pi} \left[\int_0^{\pi} \left[\int_0^{a} 1\cdot\rho^2\sin\phi\, d\rho \right] d\phi \right] d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\pi} \left[\tfrac{1}{3}\rho^3\sin\phi \right]_0^a d\phi \right] \partial\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\pi} \frac{1}{3} a^3 \sin\phi \, d\phi \right] d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} a^3 \cos\phi \right]_0^{\pi} d\theta = \int_0^{2\pi} \frac{2}{3} a^3 \, d\theta$$

$$= \frac{2}{3} a^3 \cdot 2\pi = \frac{4\pi}{3} a^3$$