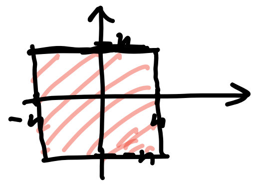
Seksjon 6.8 Integrasjon over ubegrensede områder

Definer $K_n = \{(x,y) \in \mathbb{R}^2 : |x| \leq n, |y| < n\}$



Definisjon 6.8

La $A \subseteq \mathbb{R}^2$ være slik at $A \cap K_n$ er Jordanmålbar for alle n (A kan være ubegrenset). Hvis $f : A \to \mathbb{R}$ er ikkenegativ og kontinuerlig definerer vi

$$\iint_A f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{A\cap K_n} f(x,y)\,dx\,dy$$

hvis grenseverdien eksisterer. Vi sier da at (det uegentlige integralet) $\iint_A f(x,y)\,dx\,dy$ konvergerer.

Ellers sier vi at det divergerer

Setning 6.8.3: Kan bytte ut K_n med $B(0,n)$

Definisjon 6.8.5

La A være som over

Hvis f er begrenset og kontinuerlig definerer vi

$$f_+(x,y) = \begin{cases} f(x,y) & \text{hvis } f(x,y) \geq 0 \\ 0 & \text{ellers} \end{cases} \qquad f_-(x,y) = \begin{cases} -f(x,y) & \text{hvis } f(x,y) < 0 \\ 0 & \text{ellers} \end{cases}$$

Vi har da at $f(x,y) = f_+(x,y) - f_-(x,y)$

Vi sier at $\iint_A f(x,y)\,dx\,dy$ konvergerer dersom begge

$$\iint_A f_+(x,y)\,dx\,dy \quad \text{og} \quad \iint_A f_-(x,y)\,dx\,dy \quad \text{konvergerer.}$$

I så fall definerer vi

$$\iint_A f(x,y)\,dx\,dy = \iint_A f_+(x,y)\,dx\,dy - \iint_A f_-(x,y)\,dx\,dy$$

Eksempel 1

Hva blir $\displaystyle\iint_{\mathbb{R}^2 \setminus B(0,1)} \frac{1}{(x^2+y^2)^{3/2}}\,dx\,dy$?

differens.

Løsning:

$$\iint_{\mathbb{R}^2 \setminus B(0,1)} \frac{1}{(x^2+y^2)^{3/2}}\,dx\,dy = \lim_{n\to\infty} \iint_{B(0,n)\setminus B(0,1)} \frac{1}{(x^2+y^2)^{3/2}}\,dx\,dy$$

i polarkoordinater: $0 \le \theta \le 2\pi$, $1 \le r \le n$

$$= \lim_{n\to\infty} \int_0^{2\pi} \Big[\int_1^n \frac{1}{(r^2)^{3/2}} \overset{|J|}{r}\,dr \Big] d\theta =$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \Big[\int_1^n \frac{1}{r^2}\,dr \Big] d\theta = \lim_{n\to\infty} \int_0^{2\pi} \Big[-\frac{1}{r} \Big]_1^n d\theta$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \Big(1 - \frac{1}{n}\Big) d\theta = \lim_{n\to\infty} 2\pi\Big(1 - \frac{1}{n}\Big) = \underline{\underline{2\pi}}$$

$(x,y) = \vec{T}(r,\theta) = (r\cos\theta, r\sin\theta)$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Eksempel 2

: Vis at $\displaystyle I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1$

$\frac{1}{\sqrt{2\pi}} e^{-x^2}$ er sannsynlighetstetthheten for en Gaussfordeling.

Løsning: Vanskelig å regne ut I direkte. Vi ser på dette i to dimensjoner

$$\iint_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \lim_{n\to\infty} \iint_{K_n} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$$

$$= \lim_{n\to\infty} \int_{-n}^{n} \Big[\int_{-n}^{n} \underbrace{e^{-\frac{x^2+y^2}{2}}}_{e^{-x^2/2}\,e^{-y^2/2}}\,dx \Big] dy$$

$$= \lim_{n\to\infty} \int_{-n}^{n} e^{-y^2/2} \Big[\int_{-n}^{n} e^{-x^2/2}\,dx \Big] dy$$

$$= \lim_{n\to\infty} \int_{-n}^{n} e^{-y^2/2}\,dy \int_{-n}^{n} e^{-x^2/2}\,dx = \lim_{n\to\infty} \left(\int_{-n}^{n} e^{-x^2/2}\,dx \right)^2$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx \right)^2 = \left(\sqrt{2\pi}\, I \right)^2 = 2\pi I^2 \qquad (*)$$

Kan også regnes ut ved å bytte ut K_n med $B(0,n)$, og bruke polarkoordinater:

$$\iint_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \lim_{n\to\infty} \iint_{B(0,n)} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$$

$B(0,n)$: $0 \le \theta \le 2\pi$, $0 \le r \le n$

$$= \lim_{n\to\infty} \int_0^{2\pi} \left[\int_0^{n} e^{-r^2/2}\, \underbrace{r}_{|J|}\,dr \right] d\theta$$

$u = -r^2/2$
$du = -r\,dr$
$r\,dr = -du$
$r = 0 \Rightarrow u = 0$
$r = n \Rightarrow u = -n^2/2$

$$= \lim_{n\to\infty} \int_0^{2\pi} \left[\int_0^{-n^2/2} -e^{u}\,du \right] d\theta$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \left(1 - e^{-n^2/2} \right) d\theta$$

$$= \lim_{n\to\infty} 2\pi \left(1 - e^{-n^2/2} \right) = 2\pi \qquad (**)$$

Sammenligner $(*)$ og $(**)$:

$$2\pi I^2 = 2\pi$$
$$\Updownarrow$$
$$I^2 = 1$$
$$\Updownarrow$$
$$I = \pm 1, \quad \text{slik at } I = 1.$$

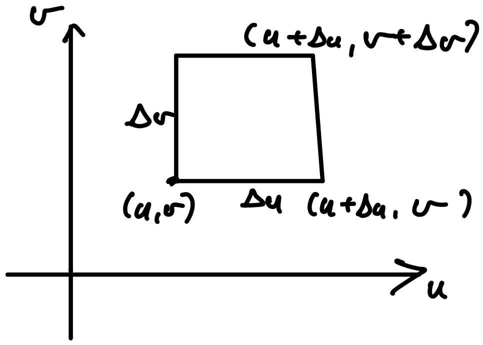
Seksjon 6.4 Flateintegraler

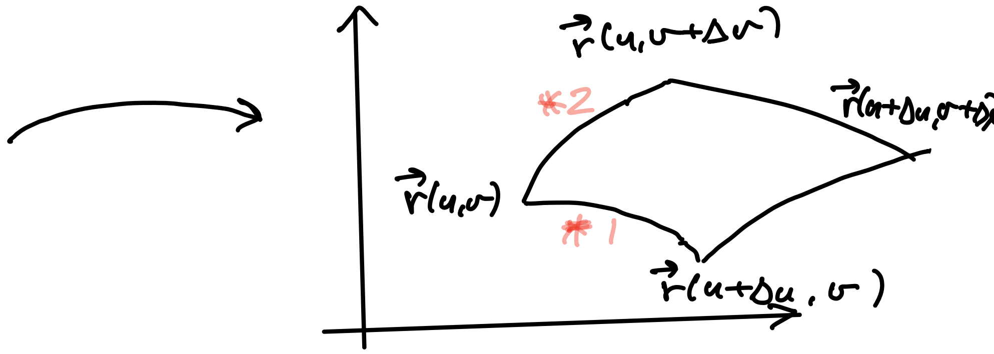
Anta T er flaten parametrisert ved

$$\vec{r}(u,v) = \left(X(u,v),\, Y(u,v),\, Z(u,v) \right) \quad (u,v) \in A.$$

Anta at f er kontinuerlig på T (tenk på som massetetthetsfunksjon)
Vi vil definere et flateintegral slik at det svarer til massen til flaten.

(Hvis $\vec{r}: \mathbb{R}^2 \to \mathbb{R}^2$: Vet at $\left| \frac{\partial \vec{r}}{\partial u} \quad \frac{\partial \vec{r}}{\partial v} \right|$ er en forstørrelsesfaktor for arealer.)





$*1 \approx \vec{r}(u+\Delta u, v) - \vec{r}(u,v) \approx \frac{\partial \vec{r}}{\partial u}(u,v)\Delta u$

$*2 \approx \vec{r}(u, v+\Delta v) - \vec{r}(u,v) \approx \frac{\partial \vec{r}}{\partial v}(u,v)\Delta v$

Arealet av bildeflaten $\approx \left| \frac{\partial \vec{r}}{\partial u}(u,v)\Delta u \times \frac{\partial \vec{r}}{\partial v}(u,v)\Delta v \right|$

$$= \left| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right| \Delta u \Delta v$$

massen av bildeflaten $\approx f(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right| \Delta u \Delta v$

Lar vi Δu og $\Delta v \to 0$, og summerer over alle små flater, så får vi at massen til flaten blir

$$\iint_A f(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right| du\, dv$$

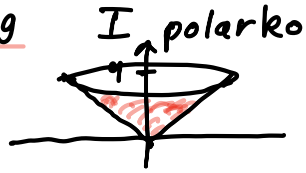
Vi kaller dette for flateintegralet til f over T, og skriver

$$\iint_T f \, dS$$

Eksempel 3 La T være flaten $z = \sqrt{x^2+y^2}$, $0 \le z \le 4$ $(0 \le r \le 4)$

Hva blir $\iint_T (x+y)\, dS$?

Løsning I polarkoordinater: $\vec{r}(r,\theta) = (r\cos\theta, r\sin\theta, r)$



$0 \le \theta \le 2\pi$
$0 \le r \le 4$

$$\frac{\partial \vec{r}}{\partial r} = (\cos\theta, \sin\theta, 1) \qquad \frac{\partial \vec{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \vec{i}(\sin\theta \cdot 0 - r\cos\theta)$$

$$- \vec{j}(\cos\theta \cdot 0 + r\sin\theta)$$

$$+ \vec{k}(r\cos^2\theta + r\sin^2\theta)$$

$$= -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\vec{k}$$

$$= (-r\cos\theta, -r\sin\theta, r)$$

$$\left|\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = r\sqrt{2}$$

$$\iint_T (x+y)\,dS = \int_0^{2\pi}\left[\int_0^4 (r\cos\theta + r\sin\theta) r\sqrt{2}\,dr\right] d\theta$$

$$= \int_0^{2\pi}\left[\int_0^4 (\cos\theta + \sin\theta\ r^2\sqrt{2}\,dr\right] d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{3}r^3\sqrt{2}(\cos\theta + \sin\theta)\right]_0^4 d\theta$$

$$= \int_0^{2\pi} \frac{64}{3}\sqrt{2}(\cos\theta + \sin\theta)\,d\theta \qquad = 0$$