Seksjon 6.5: Greens teorem

Viktig sammenheng mellom linjeintegraler og dobbeltintegraler.

Anta $\vec{F}(x,y) = (P(x,y), Q(x,y)) = \underbrace{P(x,y)\vec{i} + Q(x,y)\vec{j}}_{\text{notasjon fra boka}}$

er et vektorfelt i planet.

Vi har, med C en kurve i planet parametrisert ved $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $t \in [a,b]$,

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \Big(P(x(t),y(t)), Q(x(t),y(t)) \Big) \cdot (x'(t), y'(t))\, dt$$

$$= \int_a^b \big(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\big)\, dt$$

$$= \underbrace{\int_a^b P(x(t),y(t))\underbrace{x'(t)\,dt}_{dx} + \int_a^b Q(x(t),y(t))\underbrace{y'(t)\,dt}_{dy}}_{\text{Vi skriver}}$$

$$\int_C P\,dx + Q\,dy$$

Teorem 6.5.1 (Greens teorem) Anta C enkel (ikke skjærer seg selv) lukket ($\vec{r}(a) = \vec{r}(b)$) med en stykkevis glatt parametrisering $\vec{r}$.

La R være området avgrenset av C.

Anta de partielle deriverte til P og Q er kontinuerlige i et åpent område som inneholder R. Da er

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

der C er orientert mot klokka.

Greens teorem kan brukes til å regne ut arealer med linjeintegraler:

Korollar 6.5.4 $\text{Areal}(R) = \overset{(i)}{\int_{C} x\,dy} = \overset{(ii)}{-\int_{C} y\,dx} = \overset{(iii)}{\frac{1}{2}\int_{C} -y\,dx + x\,dy}$

Bevis: Sett (i): $\vec{F}(x,y) = (0, x) \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 0 = 1$

(ii): $\vec{F}(x,y) = (-y, 0) \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 + 1 = 1$

(iii): $\vec{F}(x,y) = (-\frac{1}{2}y, \frac{1}{2}x) \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} + \frac{1}{2} = 1$

Derfor blir $\int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C -y\,dx + x\,dy = \iint_R 1\,dx\,dy = \text{Areal}(R)$ ∎

Eksempel 1 La $R = B(0,1)$, $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j} = 6y\,\vec{i} + 7x\,\vec{j}$

Da er R området avgrenset av kurven C parametrisert ved

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} \qquad 0 \le t \le 2\pi$$

$P(x,y) = 6y = 6\sin t$

$Q(x,y) = 7x = 7\cos t$

$x'(t) = -\sin t$

$y'(t) = \cos t$

VS i Greens: $\int_C P\,dx + Q\,dy$

$$= \int_0^{2\pi} \underbrace{(6\sin t)}_{P} \cdot \underbrace{(-\sin t)}_{x'(t)} + \underbrace{(7\cos t)}_{Q}\underbrace{(\cos t)}_{y'(t)}\,dt$$

$\cos^2 t = 1 - \sin^2 t$

$$= \int_0^{2\pi} (-6\sin^2 t + 7\cos^2 t)\,dt$$

$\cos 2t = \cos^2 t - \sin^2 t$

$\cos 2t = 1 - 2\sin^2 t$

$\sin^2 t = \frac{1}{2}(1 - \cos 2t)$

$$= \int_0^{2\pi} (7 - 13\sin^2 t)\,dt$$

$$= 14\pi - 13\int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)\,dt$$

$$= 14\pi - 13 \cdot \cancel{2}\pi \cdot \frac{1}{\cancel{2}} + \frac{13}{2}\int_0^{2\pi} \cos 2t\,dt$$

$$= \pi + \frac{13}{4}\Big[\sin(2t)\Big]_0^{2\pi} = \underline{\pi}$$

Obs: Her er C orientert mot klokka!

HS i Greens: $\displaystyle\iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) = \iint_R (7-6)\,dx\,dy = \iint_R 1\,dx\,dy = \text{Areal}(R)$
$= \underline{\pi}$

Eller med polarkoordinater:

$$\int_0^{2\pi}\left[\int_0^1 1\cdot r\,dr\right]d\theta = \int_0^{2\pi}\left[\frac{1}{2}r^2\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta = \frac{1}{2}\cdot 2\pi = \underline{\pi}$$

Eksempel 2 Hva er arealet til ellipsen $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$?

Løsning: En parametrisering av ellipsen er $\vec{r}(t)=(a\cos t, b\sin t)$, $0\le t\le 2\pi$

Denne er orientert mot klokka.

Med $\underbrace{P(x,y) = -\frac{1}{2}y \qquad Q(x,y)=\frac{1}{2}x}_{(iii)}$: $\displaystyle\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = \iint_R 1\,dxdy$
$= \text{Areal}(R)$

$$\int_C P\,dx + Q\,dy = \int_0^{2\pi}\left(\left(-\frac{1}{2}y\right)(-a\sin t)+\frac{1}{2}x\,(b\cos t)\right)dt$$

$$= \int_0^{2\pi}\left(\left(-\frac{1}{2}b\sin t\right)(-a\sin t)+\frac{1}{2}a\cos t\,(b\cos t)\right)dt$$

$$= \int_0^{2\pi}\left(\frac{1}{2}ab\sin^2 t + \frac{1}{2}ab\cos^2 t\right)dt$$

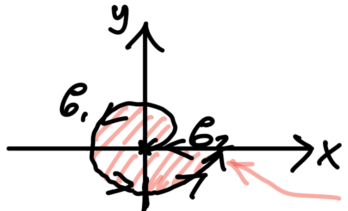
$$= \int_0^{2\pi}\frac{1}{2}ab(\sin^2 t+\cos^2 t)\,dt$$

$$= \int_0^{2\pi}\frac{1}{2}ab\,dt = \frac{1}{2}ab\cdot 2\pi = \underline{\pi ab}$$

Derfor: $\text{Areal}(R) = \pi ab.$

Eksempel 3 Hva blir arealet avgrenset av $\vec{r}(t)=(t\cos t, t\sin t)$, $0\le t\le 2\pi$, og den positive x-aksen ?

Løsning:



$(2\pi, 0)$

R er avgrenset av C_1 og C_2

(C_2 går fra $(2\pi,0)$ til $(0,0)$)

$C_1: \vec{r}_1(t) = (t\cos t, t\sin t), \quad 0 \le t \le 2\pi$

$C_2: \vec{r}_2(t) = (2\pi - t, 0) \qquad 0 \le t \le 2\pi$

$$A = \iint_R 1\,dx\,dy \overset{(?)}{=} \int_C x\,dy = \int_{C_1} x\,dy + \int_{C_2} x\,dy$$

$$\int_{C_1} x\,dy = \int_0^{2\pi} \underbrace{(t\cos t)}_{x}\overbrace{\underbrace{(\sin t + t\cos t)}_{dy}}^{y'(t)}\,dt$$

$x'(t) = \cos t - t\sin t$

$y'(t) = \sin t + t\cos t$

$$= \int_0^{2\pi} \left(\frac{1}{2} t\sin 2t + t^2\cos^2 t\right) dt$$

$$= -\frac{\pi}{2} + \int_0^{2\pi} \left(\frac{1}{2}t^2 + \frac{1}{2}t^2\cos 2t\right) dt$$

$$= -\frac{\pi}{2} + \left[\frac{t^3}{6}\right]_0^{2\pi} + \frac{1}{2}\pi$$

$$= -\frac{\pi}{2} + \frac{4}{3}\pi^3 + \frac{\pi}{2} = \frac{4}{3}\pi^3$$

$\sin 2t = 2\sin t\cos t$

$$\int_0^{2\pi} t\sin 2t\,dt = \left[-\frac{1}{2}\cos 2t \cdot t\right]_0^{2\pi} - \underbrace{\int_0^{2\pi} \left(-\frac{1}{2}\cos 2t\right)\cdot 1\,dt}_{0}$$

$$= -\frac{1}{2}\cdot 2\pi \cdot 1 = -\pi$$

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$

$$\int_0^{2\pi} t^2\cos 2t\,dt = \underbrace{\left[\frac{1}{2}t^2\sin 2t\right]_0^{2\pi}}_{0} - \int_0^{2\pi} t\sin(2t)\,dt$$

$$= \pi$$

$dy = 0$ siden $y'(t) = 0$

$$\int_{C_2} x\,dy = \int_0^{2\pi} (2\pi - t)\cdot 0\,dt = 0$$

Da blir: $\displaystyle\int_C x\,dy = \int_{C_1} x\,dy + \int_{C_2} x\,dy = \frac{4}{3}\pi^3 + 0 = \frac{4}{3}\pi^3$

Merk: Områdene våre kan ha huller:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \sum_{i=1}^{4} \int_{\mathcal{C}_i} P\,dx + Q\,dy$$

$$\mathcal{C}_2: \quad \vec{r}_2(t) = (x(t), y(t)) \qquad 0 \le t \le 1$$

$$\mathcal{C}_4: \quad \vec{r}_2(t) = (x(1-t), y(1-t)) \qquad 0 \le t \le 1$$

$$\int_{\mathcal{C}_2} P\,dx + Q\,dy = \int_0^1 P(x(t), y(t))\, x'(t) + \cdots$$

$$\int_{\mathcal{C}_4} P\,dx + Q\,dy = \int_0^1 P(x(1-t), y(1-t)) \cdot (x(1-t))'\, dt + \cdots$$

$$= \int_0^1 P(x(1-t), y(1-t)) \cdot (-x'(1-t))\, dt$$

$$u = 1-t \qquad du = -dt$$

$$= -\int_1^0 P(x(u), y(u))\,(x'(u))\,(-du)$$

$$= \int_1^0 P(x(u), y(u))\, x'(u)\, du$$

$$= -\int_0^1 P(x(u), y(u))\, x'(u)\, du$$

kansellerer.