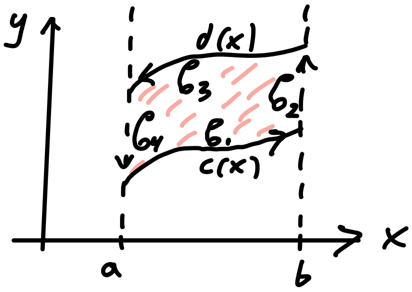
Seksjon 6.5 om Greens teorem avslutningsvis

Idé for bevis av Greens teorem for et type I-område



Vi parametriserer:

C_1 : $\vec{r}_1(t) = (t, c(t))$ $\quad a \le t \le b$

C_2 : $\vec{r}_2(t) = (b, t)$ $\quad c(b) \le t \le d(b)$

C_3 : $\vec{r}_3(t) = (t, d(t))$ $\quad a \le t \le b$ $\quad$ (motsatt vei)

C_4 : $\vec{r}_4(t) = (a, t)$ $\quad c(a) \le t \le d(a)$ $\quad$ (motsatt vei)

R : $a \le x \le b$ $\quad c(x) \le y \le d(x)$

Vi splitter opp: $\quad \displaystyle\int_C P\,dx + Q\,dy = \sum_{i=1}^{4} \int_{C_i} P\,dx + Q\,dy$

Vi får $\displaystyle\int_{C_2} P\,dx = \int_{C_4} P\,dx = 0$, siden $x = \text{konstant}$ på C_2, C_4

$\Rightarrow x'(t) = 0 \Rightarrow \underset{x'(t)dt}{\overset{\shortparallel}{dx}} = 0$

$$\int_{C_1} P\,dx \overset{\substack{x(t)=t \\ x'(t)=1}}{=} \int_a^b P(\underset{t}{x(t)}, \underset{c(t)}{y(t)})\,\underset{1}{x'(t)}\,dt = \int_a^b P(t, c(t))\,dt$$

$$\int_{C_3} P\,dx = \quad \cdots \quad = -\int_a^b P(t, d(t))\,dt$$

siden har motsatt orientering.

Vi får:
$$\int_C P\,dx = \int_a^b P(t, c(t))\,dt - \int_a^b P(t, d(t))\,dt$$

$$= -\int_a^b \Big[\underbrace{\int_{c(t)}^{d(t)} \frac{\partial P}{\partial y} P(t,y)\, dy}_{} \Big] dt$$

siden analysens fundamentalteorem sier dette er

$$-\int_a^b \Big[P(t,y) \Big]_{c(t)}^{d(t)} dt = \int_a^b P(t, c(t)) dt - \int_a^b P(t, d(t)) dt$$

Alt i alt:

$$\int_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\, dx\, dy \qquad (*)$$

På samme måte kan vi vise:

$$\int_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\, dx\, dy \qquad (**)$$

Legger vi sammen $(*)$ og $(**)$ får vi

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy,$$

som er Greens teorem.

Eksamen 2023 midtveis

1) Lineariseringen til $\vec{F}(x,y) = \begin{pmatrix} 2 + xy^4 \\ x^2 + y^2 \end{pmatrix}$ i $(1,1)$:

Løsning:

$$\vec{F}'(x,y) = \begin{pmatrix} y^4 & 4xy^3 \\ 2x & 2y \end{pmatrix} \qquad \vec{F}'(1,1) = \begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix}$$

$$\vec{F}(1,1) = \begin{pmatrix} 2+1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$T_a \vec{F}(\vec{x}) = \underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{\vec{F}(\vec{a})} + \underbrace{\begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix}}_{\vec{F}'(\vec{a})} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{a}} \right) = \begin{pmatrix} x + 4y + 3 - 5 \\ 2x + 2y + 2 - 4 \end{pmatrix}$$

$$= \begin{pmatrix} x + 4y - 2 \\ 2x + 2y - 2 \end{pmatrix}$$

2. Tangentplan til $f(x,y) = x^2 - 2x^2y + 1$ i $(1,1,0)$:

Løsning: $\frac{\partial f}{\partial x} = 2x - 4xy \qquad \frac{\partial f}{\partial y} = -2x^2$

$\frac{\partial f}{\partial x}(1,1) = -2$ $\qquad$ $\frac{\partial f}{\partial y} = -2$

$$z = f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1)$$

$$= 0 - 2(x-1) - 2(y-1)$$

$$= -2x - 2y + 2 + 2$$

$$= \underline{-2x - 2y + 4}$$

3. Hva blir $\vec{H}'(1,1,1)$ når $\vec{H}(\vec{x}) = \vec{F}(\vec{G}(\vec{x}))$,

$\vec{G}(1,1,1) = (1,1,1)$ $\quad$ $\vec{G}'(1,1,1) = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 2 & -1 & 2 \end{pmatrix}$, $\quad$ $\vec{F}'(1,1,1) = (7 \;\; 1 \;\; 3)$

?

Løsning: Kjerneregelen: $\vec{H}'(1,1,1) = \vec{F}'(\underbrace{1,1,1}_{\vec{G}(1,1,1)})\,\vec{G}'(1,1,1)$

$$= (7 \;\; 1 \;\; 3)\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 2 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 + 2 + 6 \\ 7 + 1 - 3 \\ 21 + 1 + 6 \end{pmatrix} = \underline{\begin{pmatrix} 15 \\ 5 \\ 28 \end{pmatrix}}$$

4. $A = \begin{pmatrix} -1 & 2/3 \\ -4 & 7/3 \end{pmatrix}$ $\qquad$ egenvektorer / verdier ?

Løsning: $\det(\lambda I - A) = \begin{vmatrix} \lambda + 1 & -\frac{2}{3} \\ 4 & \lambda - \frac{7}{3} \end{vmatrix} = (\lambda+1)(\lambda - \frac{7}{3}) + \frac{8}{3}$

$$= \lambda^2 - \frac{4}{3}\lambda + \frac{1}{3} = 0 : \lambda = \frac{\frac{4}{3} \pm \sqrt{\frac{16}{9} - \frac{4}{3}}}{2}$$

$$= \frac{\frac{4}{3} \pm \sqrt{\frac{4}{9}}}{2} = \frac{\frac{4}{3} \pm \frac{2}{3}}{2}$$

$$= \frac{2}{3} \pm \frac{1}{3} \Rightarrow \lambda_1 = 1, \lambda_2 = \frac{1}{3}$$

Egenvektor $\lambda_1 = 1$: $I - A = \begin{pmatrix} 2 & -\frac{2}{3} \\ 4 & -\frac{4}{3} \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix} \Rightarrow x_1 - \frac{1}{3}x_2 = 0$

$\Rightarrow$ gen. egenvektor: $\begin{pmatrix} \frac{1}{3}x_2 \\ x_2 \end{pmatrix}$

spes. egenvektor: $\underline{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$.

Egenvektor $\lambda_2 = \frac{1}{3}$: $\frac{1}{3}I - A = \begin{pmatrix} 4/3 & -\frac{2}{3} \\ 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$

egenvektor: $x_1 - \frac{1}{2}x_2 = 0 \longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_2 \\ x_2 \end{pmatrix}$

spes egenvektor: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Oppgave 5

$$\lim_{n\to\infty} A^n x = \lim_{n\to\infty} A^n(a\vec{v}_1 + b\vec{v}_2) = a \lim_{n\to\infty} A^n \vec{v}_1 + b \lim_{n\to\infty} A^n \vec{v}_2$$

$$= a \lim_{n\to\infty} 1^n \vec{v}_1 + b \lim_{n\to\infty} \left(\frac{1}{3}\right)^n \vec{v}_2$$

$$= a\vec{v}_1 + 0 = a\vec{v}_1$$

Oppgave 6

Buelengden til $\vec{r}(t) = (\sin(2t), 5t, \cos(2t))$ $0 \le t \le 3$?

Løsning:

$$\int_0^3 \sqrt{(x_1'(t))^2 + (x_2'(t))^2 + (x_3'(t))^2}\, dt$$

$$= \int_0^3 \sqrt{(2\cos(2t))^2 + 5^2 + (-2\sin(2t))^2}\, dt$$

$$= \int_0^3 \underbrace{\sqrt{4\cos^2(2t) + 25 + 4\sin^2(2t)}}_{v(t)}\, dt = \int_0^3 \sqrt{4+25}\, dt = 3\sqrt{29}$$

Oppgave 7

akselerasjon og baneakselerasjon for $\vec{r}(t) = (\sin 2t, 5t, \cos 2t)$

Løsning: $\vec{a} = (-4\sin(2t), 0, -4\cos(2t))$

$v(t) = \sqrt{29} \implies a(t) = v'(t) = 0$

Oppgave 8

Hva blir $\int_{\mathcal{C}} f\, ds$ når $f(x,y) = x^2 y$, $\vec{r}(t) = (2t, 7t)$, $0 \le t \le 3$?

Løsning: $v(t) = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{2^2 + 7^2} = \sqrt{53}$

$f(\vec{r}(t)) = x^2 y = (2t)^2\, 7t = 28t^3$

$$\int_{\mathcal{C}} f\, ds = \int_0^3 28t^3 \sqrt{53}\, dt = \left[28 \frac{1}{4} t^4 \sqrt{53} \right]_0^3$$

$f(\vec{F}(t)) \quad v(t)$

$$= 7\sqrt{53}\ 3^4 = \underline{567\sqrt{53}}$$

Oppgave 9

$\int_C \vec{F}\cdot d\vec{r}$ når $\vec{F}(x,y) = (xy, xy^3)$

$\vec{r}(t) = (-t, 2t)$, $t \in [0,1]$

Løsning:

$$\vec{F}(\vec{r}(t)) = (xy, xy^3) = (-2t^2, -8t^4)$$

$$\vec{r}'(t) = (-1, 2)$$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = 2t^2 - 16t^4$$

$$\int_C \vec{F}\cdot d\vec{r} \quad \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt = \int_0^1 (2t^2 - 16t^4)\,dt$$

$$= \left[\frac{2}{3}t^3 - \frac{16}{5}t^5\right]_0^1$$

$$= \frac{2}{3} - \frac{16}{5} = \frac{10-48}{15} = -\frac{38}{15}$$

Oppgave 10 potensialfunksjon for $\vec{F}(x,y,z) = (y^2, 2xy+z^2, 2zy)$?

Løsning:

$$\frac{\partial\phi}{\partial x} = y^2 \Rightarrow \phi(x,y,z) = xy^2 + C_1(y,z) \quad yz^2$$

$$\frac{\partial\phi}{\partial y} = 2xy + z^2 \Rightarrow \phi(x,y,z) = xy^2 + yz^2 + C_2(x,z)$$

$$\frac{\partial\phi}{\partial z} = 2zy \Rightarrow \phi(x,y,z) = yz^2 + C_3(x,y)$$

Vi kan her sette $C_1(y,z) = yz^2$, $C_3(x,y) = xy^2$, $C_2(x,z) = 0$,
da får vi $\phi(x,y,z) = xy^2 + yz^2 + y$ er en potensialfunksjon

11) $\vec{r}(t) = (\cos t, \sin t, \cos^3 t \sin^3 t + t)$, $0 \le t \le 2\pi$
$\vec{F}$ som i oppgave 10. Hvor blir $\int_C \vec{F}\cdot d\vec{r}$?

Løsning: $\vec{F}(\vec{r}(t)) = (\sin^2 t,\ 2\cos t\sin t + (\cos^3 t \sin^3 t + t)^2,$
$2(\cos^3 t \sin^3 t + t)\sin t)$

$\vec{r}'(t) = (-\sin t, \cos t, (\cos^3 t \sin^3 t + t)')$....

$\vec{F}$ er konservativ, s.a.

$\vec{r}(0) = (1,0,0)$
$\vec{r}(2\pi) = (1,0,2\pi)$

$$\int_C \vec{F}\cdot d\vec{r} = \phi(\vec{r}(2\pi)) - \phi(\vec{r}(0))$$

$$= \phi(1,0,2\pi) - \phi(1,0,0)$$

$\phi(x,y,z) = xy^2 + yz^2 + 4$

$\phi(1,0,z) = 4$

$$= 4 - 4 = 0$$

Oppgave 12: $f(x,y) = x^4 + 2x^2y^2 + y^4$ i polarkoordinater:

Løsning: $f(x,y) = (x^2+y^2)^2 = (r^2)^2 = \underline{r^4}$

Oppgave 14

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 4 & 1 \end{pmatrix}$$

Løsning:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 0 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

x_3 fri $\Rightarrow$ uendelig mange løsninger.

Oppgave 15

$$\begin{pmatrix} 1 & 1 & 2 & b_1 \\ 1 & -2 & 3 & b_2 \\ 1 & 4 & 1 & b_3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & b_1 \\ 0 & -3 & 1 & b_2-b_1 \\ 0 & 3 & -1 & b_3-b_1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & -\frac{1}{3} & (b_1-b_2)/3 \\ 0 & 0 & 0 & b_2+b_3-2b_1 \end{pmatrix}$$

Ingen løsning når $b_2 + b_3 - 2b_1 \neq 0$.
(da blir siste søyle pivotsøyle).