Forelesning 27/10

Repetisjon med eksamensoppgaver

Eksamen 2018, oppgave 1

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{matrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{matrix}$$

a) Skriv siste radvektor som en lineær komb. av de to første

Løsning: Enkelt å se at $\vec{a}_3 = -\vec{a}_1 + \vec{a}_2$

eventuelt: Lin. avhengighetsrelasjoner mellom søyler bevares ved radreduksjoner:

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{array}{c} \begin{matrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \vec{b}_3 = -\vec{b}_1 + \vec{b}_2.$$

eventuelt: $x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ utvidet matrise: $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (samme matrise)

Rangen til A: Radrommet må ha dim ≤ 2
og vi ser lett at $\vec{a}_1$ og $\vec{a}_2$ er lin. uavhengige
$\Rightarrow$ dim(radrom) = rang = 2

Dim. nullrom til A:

Vi vet: $\dim(\mathrm{Nul}\,A) + \mathrm{rang}\,A = n = 3$

$\Rightarrow \dim(\mathrm{Nul}\,A) = 3 - \mathrm{rang}\,A = 3 - 2 = 1$

$\vec{p} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$ $\vec{z} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $W = \text{Col } A$

b) Vis at $\vec{p} \in W$

Løsning: Vi må finne $\vec{x}$ s.a. $A\vec{x} = \vec{p}$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \\ -1 & 1 & 0 & -3 \end{array}\right] \\ \vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \quad \vec{p} \end{array} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \\ 0 & 2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1/2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 & 2 \\ 0 & 1 & 1/2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A $\vec{p}$

Se matlabutskrift

Siden siste søyle ikke er en pivotsøyle, så har systemet en løsning, slik at $\vec{p} \in \text{Col } A = W$.

Vis at $\vec{z} \in W^\perp$

$$\left.\begin{aligned} \vec{z} \cdot \vec{a}_1 &= (1,-1,1)\cdot(1,0,-1) = 1-1 = 0 \\ \vec{z} \cdot \vec{a}_2 &= (1,-1,1)\cdot(1,2,1) = 1-2+1 = 0 \\ \vec{z} \cdot \vec{a}_3 &= (1,-1,1)\cdot(1,1,0) = 1-1 = 0 \end{aligned}\right\} \Rightarrow \vec{z} \in W^\perp$$

c) Med $\vec{b} = \vec{p} + \vec{z} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$, finn alle minste kvadraters løsninger av $A\vec{x} = \vec{b}$

Løsning: Siden $\vec{b} = \underbrace{\vec{p}}_{W} + \underbrace{\vec{z}}_{W^\perp}$, så vet vi at $\text{proj}_W \vec{b} = \vec{p} \; (= \hat{\vec{b}})$

$\vec{x}$ minste kvadraters løsning $\Leftrightarrow$ $A\vec{x} = \text{proj}_W \vec{b} \, (= \hat{\vec{b}}) = \vec{p}$

$$[A \quad \vec{p}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \\ -1 & 1 & 0 & -3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Generell løsning:
$$\begin{aligned} x_1 \quad\quad + \tfrac{1}{2}x_3 &= 2 \\ x_2 + \tfrac{1}{2}x_3 &= -1 \end{aligned} \quad , \; x_3 \text{ fri variabel}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - \frac{1}{2}x_3 \\ -1 - \frac{1}{2}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Eventuelt: Normalligningene:

$$A^TA = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 6 & 3 \\ 1 & 3 & 2 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} 4 \\ -6 \\ -1 \end{bmatrix}$$

$$[A^TA \quad A^T\vec{b}] = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 6 & 3 & -6 \\ 1 & 3 & 2 & -1 \end{bmatrix} \sim \cdots$$

d) $\mathbb{R}^3$, utstyrt med indreproduktet

$$\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + 2x_2y_2 + x_3y_3$$

Finn en ortogonal basis for W. Finn også $\text{proj}_W \vec{y}$, der $\vec{y} = \begin{bmatrix} 0 \\ -1 \\ -5 \end{bmatrix}$

Løsning: Gram-Schmidt : (de to første søylene i A er en basis for col A)

$$\vec{v}_1 = \vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\langle \vec{a}_2, \vec{v}_1 \rangle = \langle (1,2,1), (1,0,-1) \rangle = 2 - 1 = 1$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = 2 + 1 = 3$$

$$\vec{v}_2 = \vec{a}_2 - \frac{\langle \vec{a}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{3} \\ 2 \\ 1 + \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2 \\ 4/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Det følger at $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}$ en ortogonal basis for W.

$$\langle \vec{v}_2, \vec{v}_2 \rangle = 2 \cdot 1 + 2 \cdot 9 + 4 = 24$$

$\text{proj}_W \vec{y}$: $\langle \vec{y}, \vec{v}_1 \rangle = 0 + 0 + 5 = 5$

$$\langle \vec{y}, \vec{v}_2 \rangle = 2 \cdot 0 \cdot 1 + 2 \cdot (-1) \cdot 3 + (-5) \cdot 2 = -6 - 10 = -16$$

$$\text{proj}_W \vec{y} = \frac{\langle \vec{y}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \frac{\langle \vec{y}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 = \frac{5}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{-16}{24} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 - \frac{2}{3} \\ -2 \\ -5/3 - 4/3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

Eksamen 2017 Oppgave 2

a) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$, $W = \text{Col } A$

$\ \vec{x}_1 \quad \vec{x}_2$

Finn en ortonormal basis for W

Løsning: $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$\vec{x}_2 \cdot \vec{v}_1 = -1 + 1 + 3 = 3$

$\vec{v}_1 \cdot \vec{v}_1 = 1 + 1 + 1 = 3$

$\vec{v}_2$ $\quad \|\vec{v}_2\| = \sqrt{8} = 2\sqrt{2}$

Ortogonal basis W: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \right\}$

Ortonormal basis W: $\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Finn en QR-faktorisering av A

Løsning: Vi har:

$$\vec{x}_1 = \vec{v}_1 = \sqrt{3}\,\vec{u}_1$$

$$\vec{x}_2 = \frac{\langle \vec{x}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 + \vec{v}_2 = \underbrace{\frac{3}{3}\sqrt{3}}_{\sqrt{3}}\vec{u}_1 + 2\sqrt{2}\,\vec{u}_2$$

$$\Rightarrow \underbrace{\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & 2\sqrt{2} \end{bmatrix}}_{R}$$

b) Sett $\vec{y} = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix}$. Finn $\hat{y} = \text{proj}_W \vec{y}$

$\vec{w}_1 = \frac{1}{\sqrt{3}}(1,1,1)$
$\vec{w}_2 = \frac{1}{\sqrt{2}}(-1,0,1)$
$\vec{y}\cdot\vec{w}_1 = \frac{1}{\sqrt{3}}(4+1-8) = -\frac{3}{\sqrt{3}}$
$\vec{y}\cdot\vec{w}_2 = \frac{1}{\sqrt{2}}(-4-8) = -\frac{12}{\sqrt{2}}$

Løsning: $\text{proj}_W \vec{y} = (\vec{y}\cdot\vec{w}_1)\vec{w}_1 + (\vec{y}\cdot\vec{w}_2)\vec{w}_2$

$$= -\frac{3}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{12}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= -\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 6\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -7 \end{bmatrix}$$

Skriv $\vec{y} = \vec{y}_1 + \vec{y}_2$ der $\vec{y}_1 \in W$, $\vec{y}_2 \in W^\perp$

Løsning: Vi har $\vec{y} = \underbrace{\text{proj}_W \vec{y}}_{W} + \underbrace{(\vec{y} - \text{proj}_W \vec{y})}_{W^\perp}$ ($\vec{y}_1$, $\vec{y}_2$)

Vi bruker $\vec{y}_1 = \text{proj}_W \vec{y} = \begin{bmatrix} 5 \\ -1 \\ -7 \end{bmatrix}$

$$\vec{y}_2 = \vec{y} - \text{proj}_W \vec{y} = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix} - \begin{bmatrix} 5 \\ -1 \\ -7 \end{bmatrix} = \begin{bmatrix} 4-5 \\ 1+1 \\ -8+7 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

c) Finn β_0, β_1 slik at grafen $y = f(x) = \beta_0 + \beta_1 x$ gir minste kvadraters tilnærming til $(-1, 0.4), (1, 0.1), (3, -0.8)$.

Løsning: Design matrise : $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$

observasjonsvektor : $\begin{bmatrix} 0.4 \\ 0.1 \\ -0.8 \end{bmatrix} = \underbrace{0.1\,\vec{y}}_{\vec{u}}$ $\left(\vec{y} = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix} \text{ fra c)}\right)$

Vi leter altså etter minste kvadraters løsning av $X\vec{\beta} = 0.1\vec{y}$

Svarer til å løse $X\beta = \text{proj}_W\, 0.1\vec{y} = 0.1\, \text{proj}_W \vec{y} \overset{c)}{=} 0.1 \begin{bmatrix} 5 \\ -1 \\ -7 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.1 \\ -0.7 \end{bmatrix}$

Vi radreduserer:

$$\left[\begin{array}{cc|c} 1 & -1 & 0.5 \\ 1 & 1 & -0.1 \\ 1 & 3 & -0.7 \end{array}\right] \sim \begin{bmatrix} 1 & -1 & 0.5 \\ 0 & 2 & -0.6 \\ 0 & 4 & -1.2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0.5 \\ 0 & 1 & -0.3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.2 \\ 0 & 1 & -0.3 \\ 0 & 0 & 0 \end{bmatrix}$$

A

Ser at $\beta_1 = 0.2$, $\beta_2 = -0.3$

Eventuelt: Løs normalligningene.

Finn feilvektoren $\vec{u} - \begin{bmatrix} f(-1) \\ f(1) \\ f(3) \end{bmatrix}$

Løsning: Dette er $\vec{u} - X\vec{\beta} = \begin{bmatrix} 0.4 \\ 0.1 \\ -0.8 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0.2 \\ -0.3 \end{bmatrix}$

$$= \begin{bmatrix} 0.4 \\ 0.1 \\ -0.8 \end{bmatrix} - \begin{bmatrix} 0.5 \\ -0.1 \\ -0.7 \end{bmatrix} = \begin{bmatrix} -0.1 \\ 0.2 \\ -0.1 \end{bmatrix}$$