# Repetisjon kap. 6

- Supplementary exercises kap. 6: Oppg. 20-25
- Eksamen 2014 oppg. 1
- Eksamen 2013 oppg. 1

**Oppg. 20** Hvis $\{\vec{v}_1, \dots, \vec{v}_p\}$ er en ortonormal mengde så er $\|c_1\vec{v}_1 + \dots + c_p\vec{v}_p\|^2 = |c_1|^2 + \dots + |c_p|^2$

**Løsning:** $p=2$: $\|c_1\vec{v}_1 + c_2\vec{v}_2\|^2 = \langle c_1\vec{v}_1 + c_2\vec{v}_2, c_1\vec{v}_1 + c_2\vec{v}_2\rangle$

$$= \langle c_1\vec{v}_1, c_1\vec{v}_1\rangle + \underbrace{\langle c_1\vec{v}_1, c_2\vec{v}_2\rangle}_{0} + \underbrace{\langle c_2\vec{v}_2, c_1\vec{v}_1\rangle}_{0} + \langle c_2\vec{v}_2, c_2\vec{v}_2\rangle$$

$$= \|c_1\vec{v}_1\|^2 + \|c_2\vec{v}_2\|^2$$

$$= |c_1|^2 + |c_2|^2 \quad (\text{siden } \|\vec{v}_1\| = \|\vec{v}_2\| = 1)$$

Pythagoras: $\|c_1\vec{v}_1 + c_2\vec{v}_2\|^2 = \|c_1\vec{v}_1\|^2 + \|c_2\vec{v}_2\|^2 = |c_1|^2 + |c_2|^2$

Anta vi har vist dette for $p$ vektorer.

$$\|c_1\vec{v}_1 + \dots + c_{p+1}\vec{v}_{p+1}\|^2 = \|(c_1\vec{v}_1 + \dots + c_p\vec{v}_p) + c_{p+1}\vec{v}_{p+1}\|$$

ortogonale

$$= \|c_1\vec{v}_1 + \dots + c_p\vec{v}_p\|^2 + \|c_{p+1}\vec{v}_{p+1}\|^2$$

induksjon:

$$= |c_1|^2 + \dots + |c_p|^2 + |c_{p+1}|^2.$$

$\Rightarrow$ Påstanden er også sann for $p+1$ vektorer.

**Oppg 21** Vis: Hvis $\{\vec{v}_1, \dots, \vec{v}_p\}$ ortonormal mengde så er
$$\|\vec{x}\|^2 \geq |\vec{x}\cdot\vec{v}_1|^2 + \dots + |\vec{x}\cdot\vec{v}_p|^2$$

**Løsning:** Utvid $\{\vec{v}_1, \dots, \vec{v}_p\}$ til en ortonormal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$.

Da er $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n \qquad c_i = \vec{x}\cdot\vec{v}_i$

Fra oppgave 20:

$$\|\vec{x}\|^2 = |c_1|^2 + \cdots + |c_n|^2$$
$$= |\vec{x}\cdot\vec{v}_1|^2 + \cdots + |\vec{x}\cdot\vec{v}_n|^2$$
$$= |\vec{x}\cdot\vec{v}_1|^2 + \cdots + |\vec{x}\cdot\vec{v}_p|^2 + |\vec{x}\cdot\vec{v}_{p+1}|^2 + \cdots + |\vec{x}\cdot\vec{v}_n|^2$$
$$\geq |\vec{x}\cdot\vec{v}_1|^2 + \cdots + |\vec{x}\cdot\vec{v}_p|^2$$

**Oppgave 22** Vis at $\{U\vec{v}_1, \ldots, U\vec{v}_n\}$ er en ortonormal basis for $\mathbb{R}^n$ når $\{\vec{v}_1, \ldots, \vec{v}_n\}$ er det. $U$ ortogonal og $n\times n$.

**Løsning:** $$(U\vec{v}_i)\cdot(U\vec{v}_j) = (U\vec{v}_i)^T U\vec{v}_j = \vec{v}_i^{\,T} U^T U \vec{v}_j = \vec{v}_i^{\,T}\vec{v}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j. \end{cases}$$

**Oppgave 23** Anta $(U\vec{x})\cdot(U\vec{y}) = \vec{x}\cdot\vec{y}$ for alle $\vec{x},\vec{y}$ i $\mathbb{R}^n$. Vis at $U$ er ortogonal.

**Løsning:** $$(U^TU)_{ij} = \vec{e}_i^{\,T} U^T U \vec{e}_j = (U\vec{e}_i)^T U\vec{e}_j = (U\vec{e}_i)\cdot(U\vec{e}_j) = \vec{e}_i\cdot\vec{e}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j. \end{cases}$$

$\Rightarrow U^TU = I.$ (gjelder også motsatt vei).

**Oppgave 24** Vis at egenverdiene til en ortogonal matrise er $\pm 1$:

**Løsning:** Anta $U\vec{x} = \lambda\vec{x}$.

Da er $$(U\vec{x})\cdot(U\vec{x}) = \vec{x}\cdot\vec{x} = \|\vec{x}\|^2$$
$$= (\lambda\vec{x})\cdot(\lambda\vec{x}) = \lambda^2\|\vec{x}\|^2$$

Del med $\|\vec{x}\|^2$: $1 = \lambda^2 \Leftrightarrow \lambda = \pm 1.$

**Oppgave 25** Anta $\|\vec{u}\|=1$. Vis at $Q=I-2\vec{u}\vec{u}^T$ er ortogonal.

**Løsning:** $Q^TQ = (I-2\vec{u}\vec{u}^T)^T(I-2\vec{u}\vec{u}^T)$

$$= (I-2\vec{u}\vec{u}^T)(I-2\vec{u}\vec{u}^T)$$

$$= I - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4\vec{u}\underbrace{\vec{u}^T\vec{u}}_{1}\vec{u}^T$$

$$= I - 4\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T = I$$

Vis også at $Q\vec{v} = -\vec{v}$ for $\vec{v}\in \operatorname{Span}\{\vec{u}\}$ *

$Q\vec{v} = \vec{v}$ for $\vec{v}\in (\operatorname{Span}\{\vec{u}\})^\perp$ **

**Løsning:** * Det er nok å vise at $Q\vec{u} = -\vec{u}$:

$$Q\vec{u} = (I-2\vec{u}\vec{u}^T)\vec{u} = \vec{u} - 2\vec{u}\underbrace{\vec{u}^T\vec{u}}_{1}$$

$$= \vec{u} - 2\vec{u} = -\vec{u}.$$

** Anta $\vec{v}\perp\vec{u}$

$$Q\vec{v} = (I-2\vec{u}\vec{u}^T)\vec{v} = \vec{v} - 2\vec{u}\underbrace{\vec{u}^T\vec{v}}_{0} = \vec{v}$$

**Eksamen 2014 Oppgave 1**

Sett
$$\left.\begin{aligned}\vec{a}_1 &= (1,0,0,-1)\\ \vec{a}_2 &= (1,2,0,-1)\\ \vec{a}_3 &= (3,1,1,-1)\end{aligned}\right\}\text{ lin uavh.}$$

a) Finn en ortonormal basis for $W=\operatorname{Span}\{\vec{a}_1,\vec{a}_2,\vec{a}_3\}$

**Løsning:** $\vec{v}_1 = \vec{a}_1$

$$\vec{v}_2 = \vec{a}_2 - \frac{\vec{a}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1$$

$$= \begin{bmatrix}1\\2\\0\\-1\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\2\\0\\0\end{bmatrix}$$

$\vec{v}_1\cdot\vec{v}_1 = 1+1 = 2$

$\vec{a}_2\cdot\vec{v}_1 = 1+1 = 2$

$$\vec{v}_3 = \vec{a}_3 - \frac{\vec{a}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{a}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$\vec{v}_2\cdot\vec{v}_2 = 4$

$\vec{a}_3\cdot\vec{v}_1 = 3+1 = 4$

$\vec{a}_3\cdot\vec{v}_2 = 2$

$$= \begin{bmatrix}3\\1\\1\\-1\end{bmatrix} - \frac{4}{2}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}0\\2\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}3\\1\\1\\-1\end{bmatrix} + \begin{bmatrix}-2\\0\\0\\2\end{bmatrix} + \begin{bmatrix}0\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} \qquad \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \qquad \vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$$

ortonormal basis: $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} \right\}$

b) $\vec{b} = (3,2,-2,-1)$ og vi ser på systemet

$$\underbrace{\begin{bmatrix}1 & 1 & 3\\0 & 2 & 1\\0 & 0 & 1\\-1 & -1 & -1\end{bmatrix}}_{\vec{a}_1\ \vec{a}_2\ \vec{a}_3;\ A} \underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix}3\\2\\-2\\-1\end{bmatrix}}_{\vec{b}}$$

Minste kvadraters løsning?

projeksjonen av $\vec{b}$ ned på $W$:

$\vec{b}\cdot\vec{u}_1 = \frac{4}{\sqrt{2}}$

$\vec{b}\cdot\vec{u}_2 = 2$

$\vec{b}\cdot\vec{u}_3 = 0$

$$\hat{\vec{b}} = (\vec{b}\cdot\vec{u}_1)\vec{u}_1 + (\vec{b}\cdot\vec{u}_2)\vec{u}_2 + (\vec{b}\cdot\vec{u}_3)\vec{u}_3$$

$$= \frac{4}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\\-2\end{bmatrix} + \begin{bmatrix}0\\2\\0\\0\end{bmatrix} = \begin{bmatrix}2\\2\\0\\-2\end{bmatrix}$$

Minste kvadraters løsninger av $A\vec{x} = \vec{b}$ er løsninger av $A\vec{x} = \hat{\vec{b}} = \begin{bmatrix}2\\2\\0\\-2\end{bmatrix}$

$$\begin{array}{c}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -2 \end{array}\right] \\ \underbrace{\qquad\qquad\qquad}_{A}\ \underbrace{\quad}_{\vec{b}}\end{array} \sim \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = 1 \\ x_2 = 1 \\ x_3 = 0 \end{array}$$

$\Rightarrow \vec{x} = (1,1,0)$ er den unike minste kvadraters løsningen.

## Eksamen 2013 Oppgave 1

a) $C = \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

søyle 1-3 er pivotsøyler, så rangen er 3

basis for nullrommet: $\vec{x} \in \mathrm{Nul}\, A$ ($x_4$ fri variabel)

$$\Updownarrow$$

$$\begin{aligned} x_1 \qquad + x_4 &= 0 \\ x_2 \quad + x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 \\ -x_4 \\ -x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

slik at $\left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ er en basis for nullrommet.

b) Ortogonal basis for ~~søy~~lerommet til $A$:

$$A = \begin{array}{c}\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \\ \begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{array}\end{array}$$

$$\vec{v}_1 = \vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 2/3 \\ -\frac{1}{3} \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

$\vec{v}_1 \cdot \vec{v}_1 = 3$
$\vec{a}_2 \cdot \vec{v}_1 = 0$

$\vec{a}_3 \cdot \vec{v}_1 = 1$
$\vec{a}_3 \cdot \vec{v}_2 = -1 + 1 = 0$
$\vec{v}_2 \cdot \vec{v}_2 = 3$

Vi kan sette $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix}}_{\vec{v}_3} \right\}$

c) Vi setter $\vec{y} = \begin{bmatrix} -2 \\ 0 \\ 3 \\ 2 \end{bmatrix}$, og $\hat{\vec{y}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

$$\vec{y} - \hat{\vec{y}} = \begin{bmatrix} -2 \\ 0 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

Vi har at $\hat{\vec{y}} \cdot (\vec{y} - \hat{\vec{y}}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ 3 \\ 1 \end{bmatrix} = -1 + 1 = 0$

Det følger at $\hat{\vec{y}} = \text{proj}_W \vec{y}$.

Evt kan vi regne ut

$$\frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 + \frac{\vec{y} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} \vec{v}_3$$

$$= \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{15} \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/3 - 1/3 \\ 1/3 + 2/3 \\ 1/3 - 1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \hat{\vec{y}}.$$

$$= \frac{1}{3} \vec{v}_1 + \frac{1}{3} \vec{v}_3$$

$\vec{y} \cdot \vec{v}_1 = -2 + 3 = 1$
$\vec{y} \cdot \vec{v}_2 = -2 + 2 = 0$
$\vec{y} \cdot \vec{v}_3 = 2 + 3 = 5$
$\vec{v}_3 \cdot \vec{v}_3 = 1 + 4 + 1 + 9 = 15$

Minste kvadraters løsninger til $A\vec{x}=\vec{y}$ fås ved å løse $A\vec{x}=\hat{\vec{y}}$:
Vi ser at $\hat{\vec{y}}$ er lik tredje søyle i $A$, slik at $\vec{x}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$
er en minste kvadraters løsning.

Andre løsninger: $\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + \text{Nul}\,A = \underline{\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}}$

d) Utvid $B$ til en ortogonal basis for $\mathbb{R}^4$.

Løsning: Vi så at $\vec{y}-\hat{\vec{y}} = \begin{bmatrix}-2\\-1\\3\\1\end{bmatrix}$ stod ortogonalt på Col $A$.

Det følger at $\left\{ \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\-1\\3\end{bmatrix}, \begin{bmatrix}-2\\-1\\3\\1\end{bmatrix} \right\}$ er en ortogonal basis for $\mathbb{R}^4$.

Finn koordinatene til $\vec{y}$ i denne basisen.

Løsning:

$$\begin{bmatrix} 1 & 1 & -1 & -2 & -2 \\ 1 & -1 & 2 & -1 & 0 \\ 1 & 0 & -1 & 3 & 3 \\ 0 & 1 & 3 & 1 & 2 \end{bmatrix} \sim \cdots$$

Alternativt:

Siste komponent må være 1 siden $\vec{y} = \hat{y} + \underbrace{(y-\hat{y})}_{\begin{bmatrix}-2\\-1\\3\\1\end{bmatrix}}$

For å finne de tre første komponentene: Løs

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 2 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} \\ \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \hat{\vec{y}} \end{matrix} \end{array} \sim \cdots$$

Alternativt: Fant i c) at $\hat{\vec{y}} = \frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_3$

slik at $\vec{y} = \hat{\vec{y}} + (\vec{y}-\hat{\vec{y}}) = \frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_3 + \vec{v}_4$

$\Rightarrow \underline{[\vec{y}]_B = (\frac{1}{3}, 0, \frac{1}{3}, 1)}$