Eksamen 2019

Oppgave 1

$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}$ $\quad W = \mathrm{Col}\, A$

a) Finn en basis for W

Løsning: $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Søyle 1 og 2 er pivotsøyler, slik at

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ er en basis for Col A.

Finn en basis for Nul A

ser at $\vec{x} \in \mathrm{Nul}\, A \iff \left.\begin{array}{l} x_1 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{array}\right\}$ $\quad \begin{array}{l} x_1 = -2x_3 \\ x_2 = -x_3 \end{array}$

$\Longrightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ $\quad (x_3 \text{ fri})$

Derfor er $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$ en basis for Nul A.

b) Finn en ortogonal basis for W

Løsning: Sett $\vec{v}_1 = \vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_1 = 4$

$\vec{a}_2 \cdot \vec{v}_1 = 4$

$\vec{v}_2 = \vec{a}_2 - \dfrac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$

$= \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Derfor er $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ en ortogonal basis for Col A.

c) $\vec{y} = \begin{bmatrix} 0 \\ -2 \\ 10 \\ 0 \end{bmatrix}$. Finn $\text{proj}_W \vec{y}$.

$\vec{y}\cdot\vec{v}_1 = -2+10 = 8$

$\vec{y}\cdot\vec{v}_2 = -2$

$\vec{v}_2\cdot\vec{v}_2 = 2$

Løsning: $\text{proj}_W \vec{y} = \dfrac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \dfrac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$

$$= \frac{8}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{-2}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 \\ 1 \\ 2 \\ 2 \end{bmatrix}}}$$

Finn alle minste kvadraters løsninger av $A\vec{x} = \vec{y}$

Løsning: Vi må løse $A\vec{x} = \text{proj}_W \vec{y}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 1 & 2 & 4 & 1 \\ 1 & 1 & 3 & 2 \\ 1 & 1 & 3 & 2 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underbrace{}_{A}$ $\underbrace{}_{\vec{y}}$

$\vec{x}$ må tilfredsstille: $x_1 + 2x_3 = 3$, $x_2 + x_3 = -1$ x_3, fri

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3-2x_3 \\ -1-x_3 \\ x_3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}}}$$

Evt., bruk normalligningene:

$$A^TA = \begin{bmatrix} 4 & 4 & 12 \\ 4 & 6 & 14 \\ 12 & 14 & 38 \end{bmatrix} \qquad A^T\vec{y} = \begin{bmatrix} 8 \\ 6 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} A^TA & A^T\vec{y} \end{bmatrix} = \begin{bmatrix} 4 & 4 & 12 & 8 \\ 4 & 6 & 14 & 6 \\ 12 & 14 & 38 & 22 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

d) Vi utstyrer $\mathbb{R}^4$ med indreproduktet

$$\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + 2x_2y_2 + x_3y_3 + x_4y_4$$

Finn en ortogonal basis for W med dette indreproduktet

Løsning: $\vec{v}_1 = \vec{a}_1$

$$\vec{v}_2 = \vec{a}_2 - \frac{\langle \vec{a}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1$$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\langle \vec{v}_1, \vec{v}_1 \rangle = 2 + 2 + 1 + 1 = 6$

$\langle \vec{a}_2, \vec{v}_1 \rangle = 4 + 1 + 1 = 6$

ser at $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ er en ortogonal basis også nå.

Finn $\hat{y} \in W$ s.a. $\|\vec{y} - \hat{y}\|$ er minst mulig.

nytt indreprodukt.

$$\text{proj}_W \vec{y} = \frac{\langle \vec{y}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 + \frac{\langle \vec{y}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}\vec{v}_2$$

$$= \frac{6}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{-4}{4}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}}}$$

$\langle \vec{v}_2, \vec{v}_2 \rangle = 2 + 2 = 4$

$\langle \vec{y}, \vec{v}_1 \rangle = -4 + 10 = 6$

$\langle \vec{y}, \vec{v}_2 \rangle = -4$

$\begin{matrix} 0 & -1 \\ -2 & 1 \\ 10 & 0 \\ 0 & 0 \end{matrix}$

Oppgave 2

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

a) Vis at $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ er en egenvektor for A

Løsning:

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Derfor er $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ en egenvektor for egenverdi 4.

Finn P, D slik at $A = PDP^T$

Løsning: $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$$= (4-\lambda)((3-\lambda)^2 - 1) = (4-\lambda)(\lambda^2 - 6\lambda + 8)$$

$$= (4-\lambda)(\lambda-4)(\lambda-2) = -(\lambda-4)^2(\lambda-2)$$

$\lambda = 2$ egenverdi med multiplisitet 1

$\lambda = 4$ ——— 〃 ——— 2

Egenvektor for $\lambda = 2$:

$$A - 2I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ er en egenvektor med lengde 1.

Egenvektorer for $\lambda = 4$

$$A - 4I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 - x_3 &= 0 \\ x_2 &\text{ fri.} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

normaliserer: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ Disse er ortogonale.

Ortonormal basis av egenvektorer $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Vi kan sette $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$,

og har da $A = PDP^T$.

b) Vi skal løse $\vec{x}\,'(t) = A\vec{x}(t)$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

Løsning: Generell løsning:

egenvektorer

$$\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

egenverdier

$\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$:

$$\begin{aligned} -c_1 + c_2 \quad &= 1 \\ c_3 &= 1 \\ c_1 + c_2 \quad &= 3 \end{aligned}$$

ser at $c_3 = 1$ $\qquad 2c_2 = 4 \Rightarrow c_2 = 2 \Rightarrow c_1 = 1$

$$\Rightarrow \vec{x}(t) = \overset{c_1}{}e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \overset{c_2}{2}e^{4t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \overset{c_3}{}e^{4t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + e^{4t} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Oppgave 3

$B = \{ \underset{\vec{b}_1}{1}, \underset{\vec{b}_2}{e^x \cos x}, \underset{\vec{b}_3}{e^x \sin x} \}$ (antar lin. uavh.)

$C = \{ \underset{\vec{c}_1}{1 + e^x \cos x}, \underset{\vec{c}_2}{1 + e^x \sin x}, \underset{\vec{c}_3}{e^x(\cos x + \sin x)} \}$

a) Vis at $\mathcal{C}$ er en basis for $V = \text{Span}\, \mathcal{B}$.

Finn også koordinatskiftematrisene $P_{\mathcal{B}\leftarrow\mathcal{C}}$, $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

Løsning: Vi har

$$\left.\begin{aligned} \vec{c}_1 &= \vec{b}_1 + \vec{b}_2 \\ \vec{c}_2 &= \vec{b}_1 + \vec{b}_3 \\ \vec{c}_3 &= \vec{b}_2 + \vec{b}_3 \end{aligned}\right\} \Rightarrow \left[[\vec{c}_1]_{\mathcal{B}}\ [\vec{c}_2]_{\mathcal{B}}\ [\vec{c}_3]_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Denne matrisen er inverterbar ($\det = -2$)

Da er $\mathcal{C}$ også en basis for V.

Videre er

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \left[[\vec{c}_1]_{\mathcal{B}}\ [\vec{c}_2]_{\mathcal{B}}\ [\vec{c}_3]_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$P_{\mathcal{C}\leftarrow\mathcal{B}}$:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow P_{\mathcal{C}\leftarrow\mathcal{B}} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

b) Definer $T: V \to V$ ved at

$$T(f)(x) = f'(x) \qquad \text{(Det er klart at } T \text{ er linear)}$$

Finn matrisen til T relativt til B, og relativt til C.

Løsning:

$$\begin{aligned}
& & &\text{deriverer} & & & \\
\vec{b}_1 &= 1 & &\longrightarrow & &0 & \\
\vec{b}_2 &= e^x \cos x & &\longrightarrow & &e^x \cos x - e^x \sin x &= \vec{b}_2 - \vec{b}_3 \\
\vec{b}_3 &= e^x \sin x & &\longrightarrow & &e^x \sin x + e^x \cos x &= \vec{b}_2 + \vec{b}_3
\end{aligned}$$

Det følger at $[T]_B = \left[[T(\vec{b}_1)]_B \ [T(\vec{b}_2)]_B \ [T(\vec{b}_3)]_B \right]$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[T]_C = P_{C \leftarrow B} [T]_B P_{B \leftarrow C}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 2 \\ -2 & 0 & -2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

c) Definer $S: V \to V$ ved at

$$S(f)(x) = f''(x) - 2f'(x) + 2f(x) \qquad \text{(også linear)}$$

Finn matrisen til S relativt til B.

Løsning: Vi har at $S(f) = T(T(f)) - 2T(f) + 2f$

slik at $[S]_B = ([T]_B)^2 - 2[T]_B + 2I$

Vi regner ut at $([T]_B)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$, og får

$$[S]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Alternativt: $S(\vec{b_1}) = 2$

$$S(\vec{b_2}) = e^x(\cos x - \sin x - \sin x - \cos x) - 2e^x \cos x + 2e^x \sin x + 2e^x \cos x = 0$$

$$S(\vec{b_3}) = \cdots = 0$$

$$\Rightarrow [S]_B = [\ [S(\vec{b_1})]_B\ [S(\vec{b_2})]_B\ [S(\vec{b_3})]_B\ ] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hva er $\ker(S)$?

Løsning: $Sf = 0 \iff \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} [f]_B = \vec{0}$

$\Rightarrow [f]_B$ er på formen $\begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix}$

$\Rightarrow f(x) = c_2 e^x \cos x + c_3 e^x \sin x$

$\Rightarrow \ker f = \text{Span}\{e^x \cos x, e^x \sin x\} = \text{Span}\{\vec{b_2}, \vec{b_3}\}$

Oppgave 4

a) Er matrisen $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ diagonaliserbar?

Løsning: $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix}$

$$= (4-\lambda)(\lambda^2 - 4\lambda + 4) = (4-\lambda)(\lambda-2)^2$$

Egenvektorene for $\lambda = 2$:

$$A - 2I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Ser at det er bare en fri variabel i nullrommet til $A-2I$, Det følger at egenrommet for $\lambda=2$ har dimensjon 1. Siden $\lambda=2$ er en egenverdi med multiplisitet 2 kan da ikke A være diagonaliserbar (dimensjon måtte da blitt 2).

b) $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & 4 & -4 \end{bmatrix}$

Bestem en singulærverdidekomposisjon $B = U\Sigma V^T$ av B.

Løsning: Vi regner ut $B^T B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 32 & -32 \\ 0 & -32 & 32 \end{bmatrix}$

Fra Matlab-utskrift: $B^T B$ har egenverdier: 0, 4, 64

Tilhørende egenvektorer: $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\vec{v}_3 \quad \vec{v}_2 \quad \vec{v}_1$

Vi kan sette $V = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$

$\Sigma = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (singulærverdier på diagonal)

Vi setter også

$$\vec{u}_1 = \frac{1}{\sigma_1} B\vec{v}_1 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & 4 & -4 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{8\sqrt{2}}\begin{bmatrix} 0 \\ -8 \\ -8 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} B\vec{v}_2 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & 4 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\{\vec{u}_1, \vec{u}_2\}$ kan utvides til en ortonormal basis for $\mathbb{R}^3$ ved å sette $\vec{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Får nå $A = U\Sigma V^T$ der $U = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$, $V = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$.

Max $\{ \|B\vec{x}\| : \|\vec{x}\| = 1 \}$ fås ved å velge $\vec{x} = \pm\vec{v}_1 = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$:

$$\|B\vec{x}\|^2 = \vec{x}^T \underbrace{B^T B}\vec{x}$$

Denne er symmetrisk. Bruker vi da Teorem 6, Seksjon 7.3, har vi at maks for denne er $\underbrace{M}$ = 64, som oppnås med $\vec{v}_1$ (tilh egenvektor).

største egenverdi for $B^T B$.