6.3, 6.4, 6.7 **Best approksimasjon, Gram-Schmidt prosessen**

V et indreproduktrom,
W et endeligdimensjonalt underrom.

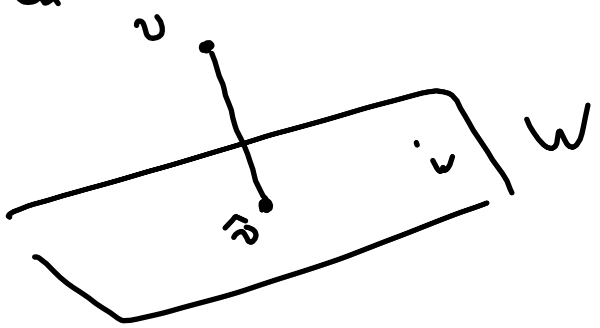
$\mathrm{Proj}_W : V \to W$,

$$v - \mathrm{Proj}_W(v) \in W^{\perp} = \{ v \in V : \langle v, w \rangle = 0 \text{ for alle } w \in W \}$$

Hvis $\{w_1, \dots, w_p\}$ er en ortogonal basis i W, da

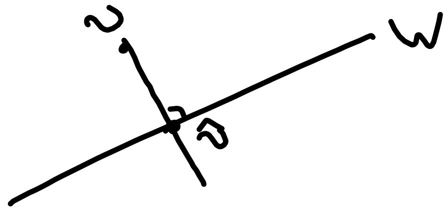
$$\mathrm{Proj}_W(v) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \dots + \frac{\langle v, w_p \rangle}{\langle w_p, w_p \rangle} w_p .$$

Teorem 9
Ta $v \in V$ og la $\hat{v} = \mathrm{Proj}_W(v)$. Da er $\hat{v}$ den vektoren i W som er nærmest v.



$\|v - \hat{v}\| \leq \|v - w\|$ for alle $w \in W$

Eks.
$\dim V = 2$, $\dim W = 1$



Bevis

Ta $w \in W$. Da

$$\|v-w\|^2 = \|\underbrace{(v-\hat{v})}_{\in W^{\perp}} + \underbrace{(\hat{v}-w)}_{\in W}\|^2$$

$$= \|v-\hat{v}\|^2 + \|\hat{v}-w\|^2 \quad \text{(Pythagoras)}$$

$$\geq \|v-\hat{v}\|^2$$

Dersom $\|v-w\| \geq \|v-\hat{v}\|$. ∎

Cauchy-Schwarz ulikheten

Hvis V er et indreproduktrom, $u, v \in V$, så

$$|\langle u,v\rangle| \leq \|u\| \cdot \|v\|.$$

Bevis 1

Anta $u = 0$. Da $\langle u,v\rangle = 0$ og $\|u\| = 0$, så ulikheten er sant.

Anta $u \neq 0$. Betrakt $W = \mathbb{R}u \subset V$,

$$\mathrm{Proj}_W : V \to W, \quad \mathrm{Proj}_W(v) = \frac{\langle v,u\rangle}{\langle u,u\rangle}u.$$

Da har vi

$$\|v\|^2 = \|\underbrace{(v-\hat{v})}_{\in W^{\perp}} + \underbrace{\hat{v}}_{\in W}\|^2 \underset{\uparrow\ \text{Pythagoras}}{=} \|v-\hat{v}\|^2 + \|\hat{v}\|^2$$

$$\geq \|\hat{v}\|^2$$

Så $\|v\| \geq \|\hat{v}\| = \left\|\frac{\langle v,u\rangle}{\langle u,u\rangle}u\right\| = \frac{|\langle v,u\rangle|}{|\langle u,u\rangle|}\|u\|$

$$= \frac{|\langle v,u\rangle|}{\|u\|^2}\|u\| = \frac{|\langle v,u\rangle|}{\|u\|}, \text{ da } |\langle v,u\rangle| \leq \|v\|\cdot\|u\|$$

Bevis 2

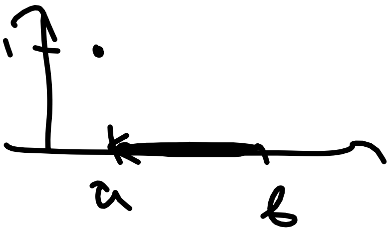
(Som ikke bruker aksiomet
$\langle u,u\rangle = 0 \implies u = 0$.)

Eks

V = reelle funksjoner $f : [a,b] \to \mathbb{R}$
slik at $\int_a^b |f(t)|^2\,dt < \infty$.

Definer $\langle f,g\rangle = \int_a^b f(t)g(t)\,dt$.

Dette tilfredsstiller alle aksiomer av indreprodukter unntatt $(\langle u,u\rangle = 0 \implies u = 0)$.



$$f(t) = \begin{cases} 1, & t = a, \\ 0, & t \in (a,b]. \end{cases}$$

Da har vi $\langle f,f\rangle = \int_a^b |f(t)|^2\,dt = 0$, men $f \neq 0$.

Definer en funksjon $f : \mathbb{R} \to \mathbb{R}$ ved

$$f(t) = \langle tu+v, tu+v\rangle.$$

Vi har at $f(t) \geq 0$ for alle $t \in \mathbb{R}$.

$$\begin{aligned} f(t) &= \langle tu+v, tu+v\rangle \\ &= t^2\langle u,u\rangle + t\langle u,v\rangle + t\langle v,u\rangle + \langle v,v\rangle \\ &= t^2\langle u,u\rangle + 2t\langle u,v\rangle + \langle v,v\rangle. \end{aligned}$$

$f(t) = t^2 \langle u,u\rangle + 2t\langle u,v\rangle + \langle v,v\rangle.$

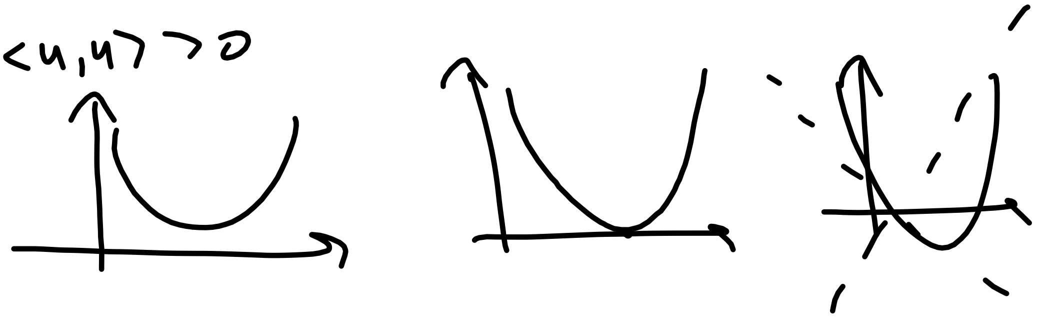
To muligheter:

1) $\langle u,u\rangle = 0.$

Siden $f(t) = 2t\langle u,v\rangle + \langle v,v\rangle \geq 0$ for alle $t \in \mathbb{R}$,

må ha $\langle u,v\rangle = 0.$

2) $\langle u,u\rangle > 0$



Vi konkluderer at $f(t)=0$ har ingen eller bare 1 reell løsning.

Dette sier hvis og bare hvis diskriminanten D til f er ≤ 0.

$p(t) = at^2 + bt + c = 0 \quad (a \neq 0)$

$t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

$D = b^2 - 4ac$ <u>diskriminanten</u> til p

$$D = (2\langle u,v\rangle)^2 - 4\langle u,u\rangle\langle v,v\rangle$$
$$= 4\left(\langle u,v\rangle^2 - \langle u,u\rangle\langle v,v\rangle\right)$$

Vi konkluderer at

$$\langle u,v\rangle^2 - \langle u,u\rangle\langle v,v\rangle \leq 0,$$

dvs. $|\langle u,v\rangle| \leq \sqrt{\langle u,u\rangle} \cdot \sqrt{\langle v,v\rangle}$

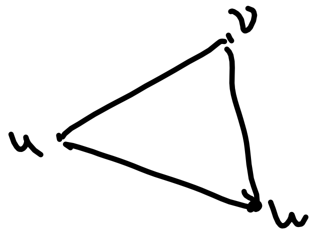
(Merk: 2) kan også håndteres som i Bevis 1) ∎

Trekantulikheten

$$\|u+v\| \leq \|u\| + \|v\|$$

($\|u-w\| = \|(u-v)+(v-w)\| \leq \|u-v\| + \|v-w\|$,

dvs. $d(u,w) \leq d(u,v) + d(v,w)$



.)

Bevis

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle \\ &\leq \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2 \\ &\underset{\text{Cauchy-Schwarz}}{\leq} \|u\|^2 + 2\|u\|\cdot\|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2, \end{aligned}$$

så $\|u+v\| \leq \|u\| + \|v\|$. ∎

Gram-Schmidt prosessen

Teorem 11

Anta at V er et indreproduktrom, $W < V$ er et endeligdimensjonalt underrom med basis $\{u_1, \dots, u_p\}$. Sett:

$$v_1 = u_1,$$
$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1,$$
$$\dots$$
$$v_k = u_k - \frac{\langle u_k, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \dots - \frac{\langle u_k, v_{k-1}\rangle}{\langle v_{k-1}, v_{k-1}\rangle} v_{k-1},$$
$$\dots$$

Da er $\{v_1, \dots, v_p\}$ en ortogonal for W.
Mer generelt, $\{v_1, \dots, v_k\}$ er en ortogonal basis for $W_k = \text{span}\{u_1, \dots, u_k\}$ for alle $k = 1, \dots, p$.

Merk:
$$v_1 = u_1,$$
$$\dots$$
$$v_k = u_k - \text{Proj}_{W_{k-1}}(u_k).$$
$$\dots$$

Bevis

Vi vil vise at $\{v_1, \dots, v_k\}$ er en ortogonal basis for W_k, $k = 1, \dots, p$.
Vi skal bruke induksjon i k.

$k=1$

Ingenting å vise: $v_1 = u_1$ er en ortogonal basis for $W_1 = \mathbb{R} u_1$.

$k \rightsquigarrow k+1$

Anta at $\{v_1, \dots, v_i\}$ er en ortogonal basis for $W_i = \text{span}\{u_1, \dots, u_i\}$ for $i = 1, \dots, k$.

$$v_{k+1} = u_{k+1} - \frac{\langle u_{k+1}, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \dots - \frac{\langle u_{k+1}, v_k \rangle}{\langle v_k, v_k \rangle} v_k$$

$$= u_{k+1} - \text{Proj}_{W_k}(u_{k+1}).$$

$v_{k+1} \neq 0$, fordi $u_{k+1} \notin W_k$.

$v_{k+1} \in W_k^{\perp}$, så $v_{k+1} \perp v_i$ for alle $i = 1, \dots, k$.

Derfor $\{v_1, \dots, v_{k+1}\}$ er et ortogonalt system.

Vektorene $v_1, \dots, v_{k+1}$ er lineært uavhengige, fordi $v_i \neq 0$ (og de er ortogonale).

Vi har

$$u_{k+1} = v_{k+1} + \text{Proj}_{W_k}(u_{k+1})$$
$$\in \text{span}\{v_{k+1}, u_1, \dots, u_k\}$$
$$= \text{span}\{v_{k+1}, v_1, \dots, v_k\}.$$

Da får vi at

$$W_{k+1} = \text{span}\{u_1, \dots, u_{k+1}\}$$
$$= \text{span}\{v_1, \dots, v_k, v_{k+1}\}.$$

Det følger at $\{v_1, \dots, v_{k+1}\}$ er en ortogonal basis for W_{k+1}. $\blacksquare$

Eksempel

$V = W = \mathbb{P}_2$, $\langle p, q \rangle = \int_0^1 p(t)\, q(t)\, dt$.

Ta standardbasisen: $p_0 = 1$, $p_1(t) = t$, $p_2(t) = t^2$

Vi bruker Gram-Schmidt prosessen for å finne en ortogonal basis for $\mathbb{P}_2$:

$$q_0 = p_0 = 1,$$

$$q_1 = p_1 - \frac{\langle p_1, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0 = t - \frac{1}{2}$$

$$q_2 = p_2 - \frac{\langle p_2, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0 - \frac{\langle p_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1$$

$$\ldots = t^2 - t + \frac{1}{6}$$

$$\langle q_0, q_0 \rangle = \int_0^1 1\, dt = 1,$$
$$\langle p_1, q_0 \rangle = \int_0^1 t\, dt = \frac{1}{2},$$
$$\langle p_2, q_0 \rangle = \int_0^1 t^2\, dt = \frac{1}{3},$$
$$\langle p_2, p_1 \rangle = \int_0^1 t^3\, dt = \frac{1}{4},$$
$$\langle p_1, p_1 \rangle = \int_0^1 t^2\, dt = \frac{1}{3},$$
$$\langle p_2, p_0 \rangle = \int_0^1 t^2\, dt = \frac{1}{3}$$

$\{1, t - \frac{1}{2}, t^2 - t + \frac{1}{6}\}$ en ortogonal basis for $\mathbb{P}_2$.