(Seksjon 13.2 forts. (Lay 6.4))

Gram - Schmidt - prosessen

V indreproduktrom

$U \subseteq V$ endeligdimensjonalt vektorrom

$B = \{\vec{b}_1, \dots, \vec{b}_k\}$ basis for U.

Da kan vi konstruere en ortonormal basis

$$B' = \{\vec{q}_1, \dots, \vec{q}_k\}$$

for U på følgende måte:

1. $\vec{m}_1 = \vec{b}_1$ og $\vec{q}_1 = \frac{1}{\|\vec{m}_1\|} \cdot \vec{m}_1$

2. $\vec{m}_2 = \vec{b}_2 - \langle \vec{q}_1, \vec{b}_2 \rangle \vec{q}_1$ og $\vec{q}_2 = \frac{1}{\|\vec{m}_2\|} \cdot \vec{m}_2$

3. $\vec{m}_3 = \vec{b}_3 - \langle \vec{q}_1, \vec{b}_3 \rangle \vec{q}_1 - \langle \vec{q}_2, \vec{b}_3 \rangle \vec{q}_2$

   og $\vec{q}_3 = \frac{1}{\|m_3\|} \cdot \vec{m}_3$

eks. 1 Finn en ortonormal basis for underrommet av $\mathbb{R}^4$ utspent av vektorene

$$\vec{b}_1 = (0,0,1,1) \;,\; \vec{b}_2 = (0,1,1,1) \;\text{ og }\; \vec{b}_3 = (1,1,1,1)$$

Løsn. Gram - Schmidt :

1. $$\vec{m}_1 = \vec{b}_1 = (0,0,1,1)$$

$$\vec{q}_1 = \frac{1}{\|\vec{m}_1\|} \cdot \vec{m}_1 = \frac{1}{\sqrt{2}} \cdot (0,0,1,1)$$

2. $$\vec{m}_2 = \vec{b}_2 - \langle \vec{q}_1, \vec{b}_2 \rangle \cdot \vec{q}_1$$

$$= (0,1,1,1) - \frac{1}{\sqrt{2}} \cdot 2 \cdot \overbrace{\frac{1}{\sqrt{2}} (0,0,1,1)}^{\vec{q}_1}$$

$$= (0,1,1,1) - 1 \cdot (0,0,1,1) = (0,1,0,0)$$

$$\vec{q}_2 = \frac{1}{\|\vec{m}_2\|} \cdot \vec{m}_2 = \frac{1}{1} \cdot \vec{m}_2 = (0,1,0,0)$$

3. $$\vec{m}_3 = \vec{b}_3 - \langle \vec{q}_1, \vec{b}_3 \rangle \vec{q}_1 - \langle \vec{q}_2, \vec{b}_3 \rangle \vec{q}_2$$

$$= (1,1,1,1) - \frac{1}{\sqrt{2}} \cdot 2 \cdot \frac{1}{\sqrt{2}} (0,0,1,1) - 1 \cdot (0,1,0,0)$$

$$= (1,1,1,1) - (0,0,1,1) - (0,1,0,0) = (1,0,0,0)$$

$$\vec{q}_3 = \frac{1}{\|\vec{m}_3\|} \cdot \vec{m}_3 = \frac{1}{1} \cdot \vec{m}_3 = (1,0,0,0)$$

Ortonormal basis : $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$

eks. 2 Finn ortonormal basis for underrommet av $\mathbb{R}^4$ utspent av

$$\vec{b}_1 = (1,2,0,2) \qquad \vec{b}_2 = (2,1,1,1) \qquad \vec{b}_3 = (1,0,1,1)$$

Løsn. Gram-Schmidt:

1 $\vec{m}_1 = \vec{b}_1 = (1,2,0,2)$

$$\vec{q}_1 = \frac{1}{\|\vec{m}_1\|}\cdot\vec{m}_1 = \frac{1}{\sqrt{9}}\cdot(1,2,0,2) = \frac{1}{3}(1,2,0,2)$$

2 $\vec{m}_2 = \vec{b}_2 - \langle\vec{q}_1,\vec{b}_2\rangle\,\vec{q}_1$

$$= (2,1,1,1) - \frac{1}{3}\cdot 6\cdot\frac{1}{3}(1,2,0,2)$$

$$= (2,1,1,1) - \frac{2}{3}\cdot(1,2,0,2)$$

$$= \frac{1}{3}\cdot(6,3,3,3) - \frac{1}{3}(2,4,0,4)$$

$$= \frac{1}{3}\cdot(4,-1,3,-1)$$

$$\vec{q}_2 = \frac{1}{\|\vec{m}_2\|}\cdot\vec{m}_2 = \frac{1}{3\sqrt{3}}(4,-1,3,-1)$$

$$\boxed{\begin{aligned}\|\vec{m}_2\| &= \frac{1}{3}\sqrt{16+1+9+1} = \frac{1}{3}\cdot\sqrt{27}\\ &= \frac{1}{3}\cdot\sqrt{9\cdot 3} = \cancel{\frac{1}{3}\cdot\sqrt{9}}\cdot\sqrt{3} = \sqrt{3}\end{aligned}}$$

Og så videre... Lykke til (oppgave til neste uke) □

$$\boxed{\begin{array}{l}\text{Bevis for Gram-Schmidt (teorem 1):}\\ \text{Viste KOLA s. 638}\end{array}}$$

Teorem 2 (QR-faktorisering)

Gitt en $(n \times k)$-matrise

$$A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_k \end{bmatrix}$$

med lineært uavhengige søylevektorer $\vec{a}_1, \ldots, \vec{a}_k \in \mathbb{R}^n$.

Hvis $\vec{q}_1, \ldots, \vec{q}_k$ er de ortonormale vektorene vi får når vi bruker Gram-Schmidt på $\vec{a}_1, \ldots, \vec{a}_k$, så har vi

$$A = \underbrace{\begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_k \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} \langle \vec{q}_1, \vec{a}_1 \rangle & \langle \vec{q}_1, \vec{a}_2 \rangle & \cdots & \langle \vec{q}_1, \vec{a}_k \rangle \\ 0 & \langle \vec{q}_2, \vec{a}_2 \rangle & \cdots & \langle \vec{q}_2, \vec{a}_k \rangle \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \langle \vec{q}_k, \vec{a}_k \rangle \end{bmatrix}}_{R}$$

Bevis Viste KOLA s. 640. Test:

$$\vec{a}_1 = \langle \vec{q}_1, \vec{a}_1 \rangle \vec{q}_1$$

$$\vec{a}_2 = \langle \vec{q}_1, \vec{a}_2 \rangle \vec{q}_1 + \langle \vec{q}_2, \vec{a}_2 \rangle \vec{q}_2$$

$$\vec{a}_3 = \langle \vec{q}_1, \vec{a}_3 \rangle \vec{q}_1 + \langle \vec{q}_2, \vec{a}_3 \rangle \vec{q}_2 + \langle \vec{q}_3, \vec{a}_3 \rangle \cdot \vec{q}_3$$

Teorem 2 sier:

$$\overbrace{\begin{bmatrix} \langle \vec{q}_1, \vec{a}_1 \rangle & \langle \vec{q}_1, \vec{a}_2 \rangle & \cdots & \langle \vec{q}_1, \vec{a}_k \rangle \\ 0 & \langle \vec{q}_2, \vec{a}_2 \rangle & \cdots & \langle \vec{q}_2, \vec{a}_k \rangle \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \langle \vec{q}_k, \vec{a}_k \rangle \end{bmatrix}}^{R}$$

$$\underbrace{\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_k \end{bmatrix}}_{Q} \quad \underbrace{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k \end{bmatrix}}_{A}$$

Ser at $\vec{a}_1 = \langle \vec{q}_1, \vec{a}_1 \rangle \cdot \vec{q}_1$

$\vec{a}_2 = \langle \vec{q}_1, \vec{a}_2 \rangle \cdot \vec{q}_1 + \langle \vec{q}_2, \vec{a}_2 \rangle \cdot \vec{q}_2$

og så videre, så det stemmer.