1 Logic

The Introduction says that a theorem is a logical consequence of a collection of axioms; within the context of those axioms it is a true mathematical statement. Our goal in this chapter is to say exactly what we mean by all of these words and to begin to see how to prove theorems.

1.1 True or False?

This section is a preliminary "thought experiment." Its goal is to help you think very explicitly about your own intuition regarding truth and falsehood in mathematics and your intuitive understanding of what is meant by proof. Then we want to build upon that understanding—to tame it, systematize it, and make it into a tool for rigorous mathematical thinking.

I recommend that you compare your own work in this section (your answers and your reasoning) with the work of a fellow student.

Thought Experiment: True or False?

Below you will find a number of mathematical assertions. Most of them deal with arithmetic, algebra, or geometry, since these subjects form a likely base of "common knowledge" for the readers of this book. They are in no particular order. Some statements are true and some are false. Sometimes you will have to assume a context that has not been spelled out for you.

Your goal is figure out the meaning of each statement and then to determine whether it is true or false. Try to justify your answers by a convincing argument, one that would convince a hardened skeptic. (You should be your own harshest critic.) Work with pencil and paper. Keep notes.

I ought to warn you that some of these statements are easy to settle, some are harder, and in one case the answer is unknown. (You are unlikely to resolve this one, but if you do, don't keep the answer to yourself!) Try not to get bogged down in any one problem. There are plenty to keep you busy.

1. The points $(-1, 1)$, $(2, -1)$, and $(3, 0)$ lie on a line.
2. If x is an integer, then $x^2 \geq x$.
3. If x is an integer, then $x^3 \geq x$.
4. For all real numbers x, $x^3 = x$.
5. There exists a real number x such that $x^3 = x$.
6. $\sqrt{2}$ is an irrational number.
7. If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.
8. If x is an even integer, then x^2 is an even integer.
9. Every positive integer is the sum of distinct powers of two.
10. Every positive integer is the sum of distinct powers of three.
11. If x is an integer, then x is even or x is odd.
12. If x is an integer, then x cannot be both even and odd.
13. Every even integer greater than 2 can be expressed as the sum of two pri numbers.
14. There are infinitely many prime numbers.
15. For any positive real number x there exists a positive real number y such t $y^2 = x$.
16. Given three distinct points in space, there is one and only one plane pass through them.

Look back over your work. You will probably find that some of your argume are sound and convincing while others are less so. In some cases you may "know" answer, but may be unable to justify it—that's OK (for now). Divide your answers i four categories: (Most students will have answers in all four categories.)

a. I am confident that the justification I gave is conclusive.
b. I am not confident that the justification I gave is conclusive.
c. I am confident that the justification I gave is *not* conclusive. (If you gave justification at all, your answer falls into this category.)
d. I could not decide whether the statement was true or false.

A number of these problems will be discussed in the coming pages, some will r But you should keep them in mind as you read. Look back over your notes from ti to time. Think about how your work in this section connects with the ideas discussec the rest of the chapter. Revise your arguments; update the information, as you can. T point is that the logical principles we are about to discuss are not completely alien to y You already have some intuition about logic, truth and falsehood, and proof. Your g in this chapter should be to incorporate the systematic treatment of logical princip into your existing understanding, sharpening insights that are right and correcting false impressions, where necessary.

1.2 Statements and Predicates

A statement is a sentence that is either true or false, but not ambiguous. For example:

- George Washington was the first President of the United States.
- Bicycles have six wheels.
- The 10^{47}th digit of π is 7.

These are all statements. Each sentence is either true or false; there is no possible ambiguity. It is not necessary that the truth or falsehood of the statement be known, only that it be unambiguous.

The following are examples of sentences that are not statements.

- *How are you doing?* It makes no sense to ask whether this sentence is true or false; questions have no truth value. Neither do imperative sentences such as "Do your homework." Only declarative sentences have truth value.
- *Picasso's* Les Demoiselles d'Avignon *is an obedient painting.* Sometimes there is no agreed-upon criterion for the truth or falsehood of a sentence. As far as I know, there is no accepted definition of "obedience" that makes sense when applied to a painting.
- *He was six feet tall.* Sometimes the sentence does not provide enough information to be unambiguously true or false. "George Washington was six feet tall," *is* a statement.

1.2.1 EXERCISE

Give some examples of sentences that are statements and some examples of sentences that are not statements. □

All of these examples bring up some very important issues. Contrast the following sentences:

- *Picasso's* Les Demoiselles d'Avignon *is an obedient painting.*
- *Picasso's* Les Demoiselles d'Avignon *is a beautiful painting.*

Most students agree that neither sentence is a statement. When asked why the first one is not, they usually say that the sentence is absurd, meaningless; therefore, asking whether it is true or false makes no sense. The second sentence is less wacky, but most still reject it as a statement because "beauty is in the eye of the beholder." The truth or falsehood of the sentence is a matter of personal opinion and is thus ambiguous.

Interestingly, a mathematician would say that the two sentences are not statements for *exactly the same reason*. The difficulty in each case lies in the lack of a careful, unambiguous definition for a key phrase—"obedient painting" in one case and "beautiful painting" in the other. As I said in the Introduction, mathematical language must describe the ideas *precisely and unambiguously*. Clear and objective definitions are essential for this. Given the variety of viewpoints on the subject of beauty, it seems impossible to find an *objective* definition of beauty on which everyone could agree—much less a sensible definition of what it means for a painting to be obedient! Without unambiguous definition, we cannot even begin to discuss the unambiguous truth or falsehood of a sentence.

Our first hint that mathematical English is closely related to, but not really the same as, colloquial English!

Sometimes there are ambiguities in the real world that we deliberately ignore in order to "capture" a subject for mathematical analysis. Consider the sentence "The Earth is not the only planet in the universe inhabited by living creatures." A biologist or a philosopher might say that whether there are living creatures on other planets is *profoundly* ambiguous. The definition of "living creature" is by no means settled and may never be. Does this mean that mathematicians (and biologists and philosophers, for that matter) cannot say anything useful about living creatures? Of course not. We use "working definitions" that are specified explicitly: "For the purposes of this study, a 'living creature' is assumed to have the following characteristics . . . " We make useful simplifying assumptions, a practice that is not only acceptable, but is an essential part of applied mathematics. We state our assumptions explicitly and accept the resulting limitations on the conclusions that we draw.

The sentence "He was six feet tall" is also too ambiguous to be a statement. However, we can interpret it as a statement by assigning a particular meaning to the term "he." We might do this by providing additional contextual information. "George Washington lived in the eighteenth century. He was six feet tall." Or we could think of such assertions as "He was six feet tall" and "x is a positive rational number" as many statements at once, one for each possible interpretation of "he" or x. In this case we might think of "he" and x as **free variables** that may be allowed to take on many possible values. For example, with the values $x = 1$, $x = \pi$, and $x = -7$, the sentence "x is a positive rational number" would become:

- 1 is a positive rational number.
- π is a positive rational number.
- -7 is a positive rational number.

These are all statements. A sentence with a free variable in it that becomes a statement when the free variable takes on a particular value is called a **predicate**. There are also predicates with two, three, or more free variables. For instance, "She went to the movies, and he went fishing," "$x^{2y} = 1$," and "$x + y = 3z$" all have more than one free variable.

1.2.2 EXERCISE

Give examples of mathematical predicates that have two and three free variables. □

Once we can recognize statements and predicates, we need to weave them together to form rigorous logical arguments. To do this, we need to analyze the logical relations between statements.

In symbolic logic, statements and their logical relations are represented by abstract typographical symbols. Statements are represented by single letters P, Q, and so on.

- P := "Beethoven wrote nine symphonies."
- Q := "*Joyful Noise* by Paul Fleishman won the 1989 Newberry Medal."
- A := "A pickle is a flowering plant."

It is convenient to borrow the familiar function notation to represent predicates having one or more free variables: T(x), R(a, b), S(someone, she), and so forth. For instance:

- T(x) := "x has wheels."
- R(a, b) := "$a > 2b$."
- S(someone, she) := "Someone said that she went to Europe in the summer of 1993."

We can make statements out of predicates by assigning values to the free variables. If T(x) is a predicate in the free variable x, and we assign the value a to x, then T(a) is a statement.

1.2.3 EXAMPLE

Suppose T(x) := "x has wheels." Then

- T(Airforce One) := "Airforce One has wheels."
- T(grass) := "Grass has wheels." ■

1.3 Quantification

We can turn a predicate into a statement by substituting particular values for its free variables. There are at least two other ways in which predicates with free variables can be used to build statements. We do this by making a claim about which values of the free variable turn the predicate into a true statement. Consider the predicate about real numbers $x^2 - 1 = 0$. Then we can write the following sentences:

- For all real numbers x, $x^2 - 1 = 0$.
- There exists a real number x such that $x^2 - 1 = 0$.

These are both statements, even though each contains the variable x without any specific meaning attached to it. The first statement is false, since $4^2 - 1 \neq 0$, and the second is true, since $(-1)^2 - 1 = 0$. The phrases **for all** and **there exists** are called **quantifiers,** and the process of using quantifiers to make statements out of predicates is called **quantification.**

In mathematical English, when we say "there exists a bludger," we don't imply that there exists *only one* bludger. We mean that there is *at least one*. If we want to say that there is *one and only one* bludger, we have to say "there exists a unique bludger."

1.3.1 EXERCISE

Suppose we understand the free variable z to refer to fish.

1. Give an example of a predicate A(z) for which "For all z, A(z)" is a true statement.
2. Give an example of a predicate B(z) for which "For all z, B(z)" is false but "There exists z such that B(z)" is true. □

"For all" is called the **universal quantifier** and "there exists . . . such that" is called the **existential quantifier.** They are so common in mathematical language that there are universally recognized symbols to represent them. The symbol for "for all" is $\forall$, the symbol for "there exists" is $\exists$, and the symbol for "such that" is $\ni$. For instance, we might say "$\forall$ positive real numbers y, y has a positive square root" or "$\exists$ a positive integer $n \ni n$ is even."

These symbols are not generally used in formal writing, so I will not use them again in the text. However, they are very convenient and are used all the time in informal mathematical discourse. My advice is to adopt them for your own use.

Quantification is so important in mathematical language that further remarks are in order. First, notice that we must take care to specify the "universe" of acceptable values for the free variables. If we are talking about real numbers, then "For all x, $x + 1 > x$" is a true statement; but if x could be Elsie the Cow, then matters are not so clear! If there is any possibility of confusion, we will have to state the range of acceptable values explicitly by saying something like "For all real numbers x, $x + 1 > x$." This can be vitally important. Compare, for instance, the statements

- For all positive real numbers x, $x > x/2$, and
- For all real numbers x, $x > x/2$.

It is essential to provide enough context to avoid ambiguities.

Second, you may find it curious that a sentence might contain a variable, as quantified statements do, and yet be a statement. The variables in statements with quantifiers are called **bound variables**. Suppose that x is a free variable taking values in the integers.

- The predicate "$x > 0$" makes an assertion about a single (but unspecified) integer x.
- The statement "For all x, $x > 0$" makes an assertion about *all* integers x, namely that they are all positive.
- Likewise, "There exists x such that $x > 0$" makes an assertion about all integers x, namely that among them there is a positive one.

The first is not a statement, the last two are. When the variables are bound, there is no ambiguity. When the variables are free, the sentence is ambiguous.

If a predicate has more than one free variable, then we can build statements by using quantifiers for each variable. The sentence "$y^2 = x$" is a predicate with two free variables, which we will suppose refer to positive real numbers. From this we could make the statement

For all x there exists y such that $y^2 = x$*!*

Note that the *order* of the quantifiers matters greatly. The statement

There exists y such that for all x, $y^2 = x$

is quite different—in fact, it is false, while the previous statement is true. (Take a moment to reflect on this; make sure you understand the difference.)

1.3.2 EXERCISE

Consider the following two statements.

1. There exists x and there exists y such that $y^2 = x$.
2. There exists y and there exists x such that $y^2 = x$.

Did quantifying over y first and then x (rather than the other way around) change the meaning of the statement? What if the quantifiers had both been "for all" instead of "there exists"? □

1.3.3 EXERCISE

Consider the predicate about integers "$x = 2y$," which contains two free variables. There are six distinct ways to use quantification to turn this predicate into a statement. (Why six?) Find all six statements and determine the truth or falsehood of each. □

It is worth noting that the phrases "for all," "for any," and "for every" are used interchangeably. Though they may convey slightly different shades of meaning in colloquial English, they all mean the same thing in mathematical English. Similarly, we might say "For some positive real number x, $x^3 - 100 > 0$" instead of "There exists some positive real number x such that $x^3 - 100 > 0$."

1.4 Mathematical Statements

We are interested in mathematics, so we will focus on mathematical statements from now on. Since we are not yet working with a specific mathematical context, I will (for the moment) discuss statements whose context and meaning you should be able to provide from your previous mathematical education. (If you occasionally run across one that you don't understand, don't worry. We are not really interested in content here, just form. Read for the general message.)

The vast majority of mathematical statements can be written in the form "If A, then B," where A and B are predicates.

But wait! If A and B are predicates involving free variables, then surely "If A, then B" is also a predicate. How do I get away with calling it a statement? In fact, by itself it is not. But it is standard practice to interpret the predicate "If A, then B" as a statement, by assuming universal quantification over the variable(s); that is, "If A(x), then B(x)" is interpreted as "For all x, if A(x), then B(x)." We will follow this convention. (I suspect that you, unknowingly, follow the convention yourself—Or did you look at virtually every statement in the "thought experiment" and argue that it was ambiguous because you didn't know the values of the free variables?) We will say more about this a little later.

1.4.1 DEFINITION

A statement in the form "If A, then B," where A and B are statements or predicates, is called an **implication.**

A is called the **hypothesis** of the statement "If A, then B." B is called the **conclusion.**

Here are some examples of implications.

1.4.2 EXAMPLE

1. If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.
2. If x is an integer, then x is either even or odd, but not both.
3. If $x^2 < 17$, then x is a positive real number.
4. If x is an integer, then $x^2 \geq x$.
5. If f is a polynomial of odd degree, then f has at least one real root. ■

1.4.3 EXERCISE

Identify the hypotheses and conclusions in each of the implications given in Example 1.4.2. □

Often mathematical statements that don't appear to be implications really are, since they can be rephrased as implications.

1.4.4 EXAMPLE

1. "$\sqrt{2}$ is an irrational number" is the same as "If $x > 0$ and $x^2 = 2$, then x is irrational."
2. "For all real numbers x, $x^3 = x$" is often written as "If x is a real number, then $x^3 = x$." ■

Most mathematical statements that are not implications are statements that assert the existence of something—in effect, predicates with existential quantification over the variables. Here are a couple of examples of existence statements.

1.4.5 EXAMPLE

1. There exists a real number x, such that $x^3 = x$.
2. There exists a line in the plane that passes through the points $(-1, 1)$, $(2, -1)$, and $(3, 0)$. ■

1.5 Mathematical Implication

Since most mathematical statements are implications (that is, they can be written in the form "if A, then B," where A and B are predicates in one or more variables) we will spend considerable time talking about them. I will begin by appealing to your intuition to motivate the definition of what it means to say that an implication is true. Then we will discuss the logical principles governing the truth and falsehood of more complicated statements. Finally we will talk about various methods of proof.

1.5.1 EXAMPLE

If x is an integer, then $x^2 \geq x$.

Proof. If $x = 0$, then $x^2 = x$, so certainly $x^2 \geq x$. The same is true if $x = 1$. If $x > 1$, then $x^2 > 1 \cdot x = x$. If $x < 0$, then $x^2 > 0 > x$. This accounts for all integer values of x. ■

What exactly did we do when we proved the theorem? We studied all values of the variable x for which the hypothesis "x is an integer" is true and showed that for those cases the conclusion "$x^2 \geq x$" is true also. We didn't consider values of x that were not integers (that is, values of x for which the hypothesis was false). We understood intuitively that those values were irrelevant to our case.

Notice that we have intuitively assumed universal quantification over the free variable.[1] To clarify, we actually proved "For all x, if x is an integer, then $x^2 \geq x$."

More generally, let us assume that A(x) and B(x) are predicates involving the free variable x. Let P(x) be the predicate "If A(x), then B(x)." In the discussion above, we considered all values of x for which A(x) is true, and we said that P(x) should be considered to be true if B(x) was true for all those values. We also said that we were uninterested in the truth value of B(x) at values of the variable x that made A(x) false, for those values of x were not relevant to the truth (or falsehood) of P(x). In fact, we dealt, in one way or another, with *all* possible values of x. We have said what it means for the *statement* "For all x, P(x)" to be true! "For all x, P(x)" is true unless there is at least one value of x for which A(x) is true and B(x) is false. That is, for specific values of x, P(x) is true unless A(x) is true and B(x) is false. In summary, for a given value of x,

$$\text{P}(x) \text{ is } \begin{cases} \text{true} & \text{if A}(x) \text{ and B}(x) \text{ are both true.} \\ \text{false} & \text{if A}(x) \text{ is true and B}(x) \text{ is false.} \\ \text{true} & \text{if A}(x) \text{ is false (regardless of the truth value of B}(x)). \end{cases}$$

1.5.2 EXERCISE

Consider now the slightly different statement "If x is an integer, then $x^3 \geq x$."

1. Show that "If x is an integer, then $x^3 \geq x$" is false.
2. Thinking in terms of hypotheses and conclusions, explain what you did to show that the statement is false. □

A value of x that makes the hypothesis A true and the conclusion B false is called a **counterexample**. In order to show that an implication is false, all we need to do is to provide *one* such example. We now see what differentiates true implications from false ones. An implication "If A(x), then B(x)" is true if B(x) is true whenever A(x) is. The implication is false if there is even one value of the variable for which the hypothesis is true and the conclusion is false.

1.5.3 EXERCISE

Occasionally you will see "If A, then B" written as "A is sufficient for B" or "B is necessary for A" or "B, if A" or "A only if B." Explain why it is sensible to say that each of these means the same thing. □

We can summarize our discussion of the truth and falsehood of implications with the following table.

[1] For simplicity, the predicates that I refer to have only one variable. Parallel statements apply to predicates with more than one variable. Quantification is assumed over all relevant variables.

A	B	*If* A, *then* B
T	T	T
T	F	F
F	T	T
F	F	T

The various lines in the table give all possible combinations of truth values for generic statements A and B. (Of course, for any specific pair of statements the truth values are determined and will therefore lie in a single line of the table.) The final column then gives the truth value for the resulting implication.

For *predicates* A(x) and B(x), "If A(x), then B(x)" is true if for all possible values of x the truth values of A and B fall only in the first, third, or fourth lines of the table. It is false if even a single value of x lands A(x) and B(x) in the second line.

An implication in which the hypothesis is false is often said to be **vacuously true.**

The term "vacuous" is used in the sense of "devoid of meaning." Sometimes statements that are vacuously true seem to us meaningless or even false. Consider, for instance, the statement

> *If the moon is made of green cheese, then chocolate prevents cavities.*

One might think that the statements "The moon is made of green cheese" and "Chocolate prevents cavities" are surely unrelated. Clearly one does not "imply" the other in everyday usage. But since the moon is not made of green cheese, the hypothesis is false, and our formal rules say that the implication is true. The moral is: Though the commonplace ideas about implication are closely related to the mathematical ones, it is important to remember that to a mathematician, implication is a specialized logical relation that need not have anything to do with cause and effect, as it does in everyday usage.

Why then did we define the truth of implications in such a peculiar way? Remember the proof of Example 1.5.1: We considered only integers x because we understood intuitively that the cases in which the hypothesis was false were irrelevant to our situtation. It would have been strange to consider the case in which x was a leopard. Such a case *could never generate a counterexample,* so the truth of the implication was not in danger from leopards.

1.6 Compound Statements and Truth Tables

Suppose that A and B are statements. More complex statements can be built from these, and we can examine their logical structure from the point of view of truth or falsehood. We have already studied "If A, then B" in detail. Symbolically, we write "If A, then B" as $A \Longrightarrow B$, which is read **"A implies B."** Recall the table that we used to summarize our discussion of implications:

A	B	$A \Longrightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

This is an example of a **truth table.** As you remember, each line of the truth table gives all possible truth values for A and B and the resulting truth value of the implication. If A and B are predicates, "If A, then B" is true if all possible values of the free variable(s) make the truth values of A and B fall in the first, third, or fourth line of the table.

In addition to implication, statements A and B can be combined in a number of ways. The most important are in the following list.

- "A and B" is called the **conjunction** of A and B. We denote it by $A \wedge B$.
- "A or B" is called the **disjunction** of A and B. We denote it by $A \vee B$.
- "Not A" is called the **negation** of A. We denote it by $\sim A$.
- "A if and only if B" is called the **equivalence** of A and B. We denote it by $A \Longleftrightarrow B$. ("If and only if" is often abbreviated iff.)

All of these are called **compound statements.** As in the case of implication, the truth values of these compound statements are defined in terms of the truth values of their individual components.

1.6.1 DEFINITION

Suppose that A and B are statements. The following truth table gives the truth values of $A \Longrightarrow B$, $A \wedge B$, $A \vee B$, $\sim A$, and $A \Longleftrightarrow B$ in terms of the truth values of A and B.

A	B	A implies B $A \Longrightarrow B$	A and B $A \wedge B$	A or B $A \vee B$	not A $\sim A$	A iff B $A \Longleftrightarrow B$
T	T	T	T	T	F	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

1.6.2 EXERCISE

Examine the preceding table. Given the colloquial meaning of the terms "and," "or," "not," and "equivalent," explain why the truth values given in the table make sense. (Note that the mathematical "or" corresponds to the colloquial "and/or." If you wish to indicate that one or the other of two statements is true *but not both,* you must say so explicitly.) □

More complex compound statements can be formed by combining conjunction, disjunction, negation, implication, and equivalence in various ways. Given the basic truth tables presented, we can find the truth tables for other compound statements.

1.6.3 EXAMPLE

Given that A and B are statements, here are the truth tables for

1. $B \wedge \sim B$:

B	~B	B ∧ ~B
T	F	F
F	T	F

Notice that the statement $B \wedge \sim B$ is *always* false regardless of the truth value for B. There is no need to check values for free variables. No choice of a free variable will ever yield a true statement of the form $B \wedge \sim B$. *(Think about what is meant by $B \wedge \sim B$. Explain why it makes sense for this statement to be false always.)* A compound statement that is always false regardless of the truth values of the simpler statements involved is called a **contradiction.**

2. $(A \wedge \sim B) \Longleftrightarrow \sim(A \Longrightarrow B)$:

A	B	~B	A ∧ ~B	A ⟹ B	~(A ⟹ B)	(A ∧ ~B) ⟺ ~(A ⟹ B)
T	T	F	F	T	F	T
T	F	T	T	F	T	T
F	T	F	F	T	F	T
F	F	T	F	T	F	T

Just as $B \wedge \sim B$ was false regardless of the truth value for B, notice that $(A \wedge \sim B) \Longleftrightarrow \sim(A \Longrightarrow B)$ is true regardless of the truth values of A and B. A compound statement that is always true is called a **tautology.** ■

You can also form compound statements involving three or more simpler statements. Work out the following example of a tautology for yourself.

1.6.4 EXERCISE

Verify that

$$(A \Longrightarrow (B \vee C)) \Longleftrightarrow ((A \wedge \sim B) \Longrightarrow C)$$

is a tautology by showing that

$$(A \Longrightarrow (B \vee C)) \quad \text{and} \quad (A \wedge \sim B) \Longrightarrow C)$$

have the same truth values.

A	B	C	$B \vee C$	$\sim B$	$A \wedge \sim B$	$A \Longrightarrow (B \vee C)$	$(A \wedge \sim B) \Longrightarrow C$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

□

Notice that truth tables for statements involving only one primitive statement have only two rows. (See part 1 of Example 1.6.3.) If there are two primitive statements (e.g., part 2 of Example 1.6.3) we use four rows. Exercise 1.6.4 involved three primitive statements and we used eight rows. These were the number of rows necessary to list all possible combinations of true and false for the primitive statements. How many rows will you need to work out a truth table for a compound statement involving four or more statements?

We could list combinations of T's and F's in any order, but then we would have to keep track of which have been listed and which not, make sure there were no repetitions, and so forth. It is convenient to have a pattern scheme that is guaranteed to give us what we want without a lot of trial and error. Look back at the examples and notice the patterns of T's and F's used to give all possible combinations of true and false for one, two (Example 1.6.3), and three statements (Exercise 1.6.4). Can you guess what pattern of T's and F's will work if there are four statements? What if there are more than four?

1.7 Learning from Truth Tables

Ultimately, truth tables are not really much good unless they teach us something. This section will, in a few sample lessons, give you a sense of what sorts of things we can learn from truth tables.

For the entire section, the capital letters A, B, and C will be understood to be predicates.

Lesson 1—Tautologies

Since a tautology is true regardless of the truth values of the underlying primitive statements, tautological statements express logical relationships that hold in any context. The following exercise contains some early lessons.

1.7.1 EXERCISE

Consider the following statements.

1. $(A \Longrightarrow (B \wedge C)) \Longrightarrow (A \Longrightarrow B)$.
2. $(A \wedge (A \Longrightarrow B)) \Longrightarrow B$.
3. $((A \Longrightarrow B) \wedge (B \Longrightarrow C)) \Longrightarrow (A \Longrightarrow C)$.

Each of these statements is a tautology and each embodies an important (and fairly intuitive) logical principle.

- Your first task is to verify that the statements are tautological by constructing truth tables for them.
- Your second task is to figure out what the logical principles are and what they tell us about proving theorems. (It will help to convert the symbols into words.) □

Lesson 2—What About the Converse?

1.7.2 DEFINITION

The implication $B \Longrightarrow A$ is called the **converse** of $A \Longrightarrow B$.

1.7.3 EXERCISE

Construct a truth table to show that it is possible for $A \Longrightarrow B$ to be true while its converse $B \Longrightarrow A$ is false, and vice versa. □

So what is the moral of this exercise? The truth of the statement "If A, then B" *does not* imply the truth of its converse. That is, knowing that A implies B *does not* tell us that B implies A.

Figure 1.1 Confusing the statement with its converse

1.7.4 EXAMPLE

Construct the converses of the following statements.

1. If Elsie is a cow, then Elsie is a mammal.
2. If $x = 0$, then $x^2 = 0$. ■

Note that "If Elsie is a cow, then Elsie is a mammal" is a true statement, whereas its converse is false. After all, Elsie might be a kangaroo, a mammal that is not a cow. Note, however, that the truth table you constructed in Exercise 1.7.3 does not go so far as to tell us that if A $\Longrightarrow$ B is true, then its converse is false. Sometimes an implication and its converse are both true, as illustrated by the example "If $x = 0$, then $x^2 = 0$."

Moral. You have to treat a statement and its converse as distinct mathematical claims, each of which requires separate verification.

1.7.5 EXERCISE

Find an example of a true statement whose converse is false and one whose converse is true. □

Lesson 3—Equivalence and Rephrasing

Consider the truth table for A $\Longleftrightarrow$ B:

A	B	$A \Longleftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Notice that $A \Longleftrightarrow B$ is true exactly when A and B have the *same truth value*. Suppose A and B are predicates and $A \Longleftrightarrow B$ is true; then we say that A and B are **equivalent**. This is because A and B are either both true or both false. Thus if we manage to prove A, we know that B is true, too. Conversely, if we prove B, we know that A is true. For all mathematical purposes we may view them as *the same statement phrased in different ways.*

Since equivalent statements are just different ways of stating the same idea, we can use truth tables to explore different ways of phrasing certain sorts of mathematical statements.

1.7.6 EXAMPLE

In Exercise 1.6.4 you showed that

$$(A \Longrightarrow (B \vee C)) \quad \text{and} \quad ((A \wedge \sim B) \Longrightarrow C)$$

are equivalent statements. In other words, any statement in the form "If A, then B or C" can be rephrased in the form "If A and not B, then C." (Of course, since "B or C" has the same meaning as "C or B," it is easy to see that "If A and not C, then B" is also equivalent.)

This is an important principle because when we want to prove a statement of the form "If A, then B or C," we usually prove one of the two equivalent forms:

- If A and not B, then C.
- If A and not C, then B. ■

1.7.7 EXERCISE

Show by constructing a truth table that

$$(A \Longleftrightarrow B) \Longleftrightarrow ((A \Longrightarrow B) \wedge (B \Longrightarrow A))$$

is a tautology. □

This truth table shows that the statement "$A \Longleftrightarrow B$" is equivalent to the conjunction of "If A, then B" and "If B, then A." That is, "$A \Longleftrightarrow B$ is true" can be rephrased by saying that "If A, then B" and its converse are both true statements.

This gives us a method for proving that two statements A and B are equivalent. We have to prove two implications. We first prove that A implies B and then that B implies A.

The phrases "A is equivalent to B" and "A if and only if B" are used interchangeably. You will also occasionally see "A is necessary and sufficient for B."

1.7.8 EXERCISE

There are some very useful rephrasings that involve negation. Construct a truth table that will allow you to compare the truth values of the following four statements.

$$\sim(A \wedge B) \qquad \sim A \wedge \sim B \qquad \sim(A \vee B) \qquad \sim A \vee \sim B$$

Which pairs are equivalent? □

We will discuss other rephrasings that involve negation later in the chapter.

1.8 Negating Statements

Consider the truth table for the negation of a statement A.

A	~A
T	F
F	T

Notice that if A is a predicate, ~A is a predicate that is true exactly when A is false and false when A is true. Thus if we manage to prove A, we know that ~A is false. Conversely, if we *disprove* A, we know that ~A is true. Thus, the negation of A is a statement of what it *means* for A to be false.

We can always write "It is not true that A" for the negation of A, but generally speaking it is more useful, when we are proving theorems, to say what *is* true rather than to say what is *not* true; negative statements do not generally tell us as much as positive statements. So it is important to be able to translate a negative statement into a positive statement. For instance, if x is a free variable taking its values in the integers it is usually preferable to restate "x is not even" as "x is odd."

1.8.1 EXERCISE

Rephrase the statement "x is not greater than 7" in positive terms. □

Though it is often possible to rephrase a negative statement as a positive statement, this is not always the case. (In which case, of course, we have to leave it in negative terms.)

Not only do we need to be able to rephrase simple statements like "x is not even" in positive terms, we need to be able to negate more complicated statements, as well. Since mathematical statements are often implications, since they have quantifiers, conjunctions, and disjunctions, we need to know how to interpret the negations of these in positive terms. The good news is that there are some general rules to help us along.

Exercise 1.7.8 told us how to negate the conjunction and disjunction of two predicates. The negation of A ∨ B is ∼A ∧ ∼B and the negation of A ∧ B is ∼A ∨ ∼B.

1.8.2 EXERCISE

Think colloquially about the meaning of AND and AND/OR. Explain why it makes sense for the negation of A ∨ B to be ∼A ∧ ∼B and for the negation of A ∧ B to be ∼A ∨ ∼B. □

1.8.3 EXERCISE

Negate the following statements. Write the negation as a positive statement, to whatever extent is possible.

1. $x + y$ is even and $y + z$ is even. (x, y, and z are fixed integers.)
2. $x > 0$ and x is rational. (x is a fixed real number.)
3. Either l is parallel to m, or l and m are the same line. (l and m are fixed lines in $\mathbb{R}^2$.)
4. The roots of this polynomial are either all real or all complex. (A complex root is, for the purposes of this exercise, one that has a nonzero imaginary part. In ordinary mathematical usage, real numbers are also complex numbers, they are just numbers whose imaginary part is zero.) □

Since quantifiers show up in mathematical statements, we must know how to negate statements containing them. As usual, to negate a statement we must decide what it means for the statement to be false. Consider the statement "All senators take bribes." Under what circumstances would this be false? In order to show it to be false, we would have to show that there is (at least) one senator who does not take bribes. The negation of the statement "All senators take bribes" is the statement "There exists some senator who does not take bribes."

1.8.4 EXERCISE

Using similar reasoning, find the negation of the statement "There exists a fast snail." □

1.8.5 EXERCISE

Negate the following statements. Write the negation as a positive statement, to whatever extent is possible.

1. There exists a line in the plane passing through the points $(-1, 1)$, $(2, -1)$, and $(3, 0)$.
2. There exists an odd prime number.
3. For all real numbers x, $x^3 = x$.
4. Every positive integer is the sum of distinct powers of three.
5. For all positive real numbers x there exists a real number y such that $y^2 = x$.
6. There exists a positive real number y such that for all real numbers x, $y^2 = x$. □

1.8.6 EXAMPLE

Sometimes it helps to deal with very complex statements more carefully, one step at a time. Consider the statement "All Martians are short and bald, or my name isn't Darth Vader."

We now consider various substatements:

- A := All Martians are short and bald.
- B := My name isn't Darth Vader.
- C := All Martians are short.
- D := All Martians are bald.

Clearly, A is equivalent to $C \wedge D$. Our original statement is $A \vee B$. So the negation of our original statement is

$$\sim(A \vee B) \iff (\sim A \wedge \sim B) \iff (\sim(C \wedge D) \wedge \sim B) \iff (\sim C \vee \sim D) \wedge \sim B.$$

We can now clearly see that the negation of "All Martians are short and bald or my name isn't Darth Vader" is "Either some Martian is tall or some Martian has hair, and my name is Darth Vader." ■

1.8.7 PROBLEM

If you want a challenge, try using this process to negate

> *You can fool some of the people all of the time, and some of the people none of the time, but you cannot fool all of the people all of the time.*

□

1.8.8 EXERCISE

Let A and B be statements. Show by constructing a truth table that the following statements are equivalent:

$$\sim(A \Longrightarrow B) \quad \text{and} \quad A \wedge \sim B.$$

□

1.8.9 EXERCISE

Negate the statement "If $x^2 > 14$, then $x < 10$." Using your intuition about this example to help you, explain why it makes sense to say that if A is true and B is false, then "If A, then B" is false. □

Negating a statement of the form "If A(x), then B(x)" has a slight wrinkle. If A(x) and B(x) are predicates, A(x) $\wedge$ ~B(x) is not a statement at all, it is a predicate, too. It seems bad to negate a statement and get a predicate! The solution to this conundrum lies in the fact that "If A, then B" is not really a statement, either. The *statement* is actually "For all x, if A(x), then B(x)." So when we negate the implication, we have to negate the quantifier, as well. The negation of the statement "For all x, A(x) $\Longrightarrow$ B(x)" is, therefore, "There exists x such that A(x) $\wedge$ ~B(x)." But following the convention that suppresses the universal quantifier in the implication, the existential quantifier in its negation is often suppressed, as well—but only if there is no possibility of confusion as a result! Absolute clarity is always the goal; if there is any ambiguity, always include the quantifier.

This is a good time to review the discussion of counterexamples on page 16. There we provided an intuitive discussion of this idea. Notice that our intuitive discussion of what it means for an implication to be false imposed an existential quantifier.

1.8.10 EXERCISE

Negate the following statements. Write the negation as a positive statement, to whatever extent is possible.

1. If x is an odd integer, then x^2 is an even integer.
2. If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.
3. If f is a continuous function, then f is a differentiable function.
4. If f is a polynomial, then f has at least one real root. □

1.8.11 EXERCISE

Let f be a function that takes real numbers as inputs and produces real numbers as outputs. Negate the following statement.

For all positive real numbers r, there exists a positive real number s such that if the distance from y to 3 is less than s, then the distance from $f(y)$ to 7 is less than r.

(*Hint:* This problem is really tricky; it will help to think carefully about quantifiers.

- The statement is of the form

  For all r there exists s such that (A(s, y) $\Longrightarrow$ B(r, y)).

  This is, in turn, of the form

  For all r there exists s such that P(r, s, y).

- Since the predicate $P(r, s, y)$ is quantified over both r and s, y is the only free variable in the implication $(A(s, y) \Longrightarrow B(r, y))$ which must, therefore, be interpreted as "For all y, $(A(s, y) \Longrightarrow B(r, y))$."
- So to negate the statement properly, we must interpret it as

For all r there exists s such that for all y, $(A(s, y) \Longrightarrow B(r, y))$.

Does the statement that you come up with for the negation coincide with your intuition about what it would mean for the statement to be false?) □

1.9 Existence Theorems

We are finally ready to discuss proof! We will start with proofs of existence. A theorem that asserts the existence of something is called an **existence theorem.** Recall the following statement from the "thought experiment." I presume that you decided the statement

There exists a real number x such that $x^3 = x$

was true. How is this demonstrated? Well, the statement asserts the existence of something, so the best way to demonstrate that it is true is simply to exhibit such an object.

$$\underbrace{\text{Consider the number 1.}}_{\text{Produce a candidate.}} \quad \underbrace{\text{Since } 1^3 = 1\text{, the statement is true.}}_{\text{Show that it does what you want.}}$$

We usually have to work harder than this to produce the object that we need, but the process of proving an existence theorem is always the same. Suppose we want to prove that "There exists a clacking waggler." We prove existence theorems in two steps.

1. We produce a "candidate." That is, we describe an object that we claim should be a clacking waggler.
2. We show that our candidate actually is what we claim it is. In this case, we show that it is a waggler and that it clacks.

One thing about existence proofs may seem baffling at first. When a candidate is produced, the proof need not tell you where the candidate came from or why it was chosen, just as a chess player need not tell you what strategy she or he used to decide what move to make. The only mathematical requirement is that the candidate be given explicitly and that the proof show the candidate does what it is meant to do.

The appearance of a candidate out of nowhere can seem a little like mathematical voodoo, especially if the choice is not an obvious one. There is a strategy operating, but it is conceived "off stage" where you don't see it. The first line of an existence proof is the end result of the reasoning process; therefore, proving existence theorems usually requires a lot of "scratch work" before the proof can be written.

Consider now the statement

There exists a line in the plane passing through the points $(-1, 1)$, $(2, -1)$, *and* $(3, 0)$.

I presume that in the "thought experiment" you decided this statement was false. For good measure, we will think through how to prove this, using some of the language we have been developing.

Saying that the statement is false is the same as saying that its negation is true. In Exercise 1.8.5 you showed that the negation of this statement is (something like)

If ℓ *is a straight line,* ℓ *fails to pass through at least one of the points* $(-1, 1)$, $(2, -1)$, *and* $(3, 0)$.

So in order to prove that the statement is false, we have to examine *every straight line in the plane* and show that at least one of the three points fails to lie on it. Starting from the notion that straight lines are of the form $\ell(x) = mx + b$, an argument might go something like this.

Proof. We start with the straight line $\ell(x) = mx + b$. If the points $(-1, 1)$ and $(2, -1)$ are to lie on the line, the following must be true:

$$1 = m(-1) + b \qquad \text{and} \qquad -1 = m(2) + b.$$

Solving these two equations simultaneously, we see that the only possibilities for m and b are $m = -2/3$ and $b = 1/3$. Therefore, the only line that has any chance of containing all three points is $\ell(x) = (-2/3)x + 1/3$. But it is not true that $0 = (-2/3)(3) + 1/3$, so $(3, 0)$ does not lie on the line. The three points do not all lie on a single line. ∎

1.10 Uniqueness Theorems

Many mathematical objects are unique. That is, there is only one of them: cube roots of real numbers, inverses of functions, solutions to differential equations (under suitable conditions).[2] You will occasionally be called upon to prove the uniqueness of a mathematical object (frequently right after you have proved its existence). Suppose you know that there is a clacking waggler, and you want to show that it is unique. That is, that there is only one clacking waggler. You do this by assuming that you have two clacking wagglers and demonstrating that they must be the same. A theorem that guarantees the uniqueness of a mathematical object is called a **uniqueness theorem**.

Contrary to popular usage, the word *unique* does not mean "distinctive" or "idiosyncratic," it means (literally) "one of a kind." Thus, if we say that some object is unique, we mean that there is only one.

1.10.1 EXAMPLE

Assume that $x^3 + 37$ has a real root. (This is true. All polynomials of odd degree have at least one real root.) Prove that it has only one.

Proof. Assume that x_1 and x_2 are real numbers and that $x_1^3 - 37 = 0$ and $x_2^3 - 37 = 0$. Then $x_1^3 - 37 = x_2^3 - 37$. So $x_1^3 = x_2^3$. Since cube roots of real numbers are unique, $x_1 = x_2$. ■

I actually proved a stronger uniqueness result. Can you see what it is?

1.11 Examples and Counterexamples

In Section 1.5 we said that in order to prove that an implication is false, we need only provide a counterexample. That is, if A and B are predicates, the statement "If A, then B" is true, unless there is some value of the variable(s) that makes A true and B false. When we provide a counterexample, we are just showing that such a value exists.[3]

1.11.1 EXERCISE

Give counterexamples to the following proposed (but false) statements.

1. If a real number is greater than 5, then it is less than 10.
2. If x is a real number, $x^3 = x$.
3. All prime numbers are odd numbers. *What is the hypothesis here? What is the conclusion?*

[2] Never underestimate a theorem that tells you that if you have one you have them all. Uniqueness theorems are very powerful. You probably use some uniqueness theorems by reflex without even thinking about them. You will run into them a lot as you continue your mathematical studies.

[3] In effect, providing a counterexample is sort of an existence proof. And it's handled pretty much the same way. Give the example and show that it makes the hypothesis true and the conclusion false.

4. If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.
5. Given three distinct points in space, there is one and only one plane passing through all three points. □

1.11.2 EXERCISE

Think again about the implications given in the "thought experiment." When you decided that one of them was false, did you justify your conclusion by means of a counterexample? If there are some you haven't justified satisfactorily, does this language help you to "fill out" the arguments? □

Counterexamples give us a straightforward procedure for showing that an implication is false. But how do we prove that an implication is true? Before we answer this question, I want to discuss a final preliminary issue. Let's consider the statement

Every positive integer is the sum of distinct powers of two.

In trying to evaluate the truth or falsehood of a theorem like this, I start by trying a lot of examples.

- $3 = 2 + 1 = 2 + 2^0$.
- $5 = 4 + 1 = 2^2 + 2^0$.
- $6 = 4 + 2 = 2^2 + 2$.
- $7 = 2^2 + 2 + 2^0$.
- $9 = 2^3 + 2^0$.

(So far, so good. Let's skip around a bit.)

- $29 = 2^4 + 2^3 + 2 + 1$.
- $113 = 2^6 + 2^5 + 2^4 + 2^0$.

(OK, so what about *really big* numbers?)

- $5{,}678{,}984 = 2^{22} + 2^{20} + 2^{18} + 2^{17} + 2^{15} + 2^{13} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^3$.

All these still wouldn't quite convince me, so I would try a bunch more. As I check more and more cases, I begin to think that the statement is probably true.

The hunch that makes me want to draw this conclusion is called **inductive reasoning.** It is the process by which we draw conclusions about the general based on the particular. (That is, we look at some examples, identify a common element, and then guess that the common element holds in all cases.) This is contrasted with **deductive reasoning,** which is the process of using the rules of logic to deduce logical consequences from assumed premises or previously proved theorems. Things can only be conclusively proved by deductive reasoning. In the preceding example, checking 100 or 1000 cases might strengthen my hunch, but it still would not prove anything conclusively.

Inductive reasoning is still important for mathematicians, of course, because it is the tool by which we make conjectures. Once you think you know what is true, you can concentrate on finding a proof. But making a guess about what is true—even a very informed guess—is simply not the same as proving it.

I'm not just being picky here. To show you how dangerous it is to assert that something is proved based on having checked even a large number of examples, consider the following example found in *Induction in Geometry* by L. I. Golovina and I. M. Yaglom.

1.11.3 EXAMPLE

Consider the polynomial $991n^2 + 1$. Suppose that you were to start evaluating this polynomial at successive positive integers at the rate of one per second. You would never get a perfect square. Not because it never *is* a perfect square, but because it would take you on the order of 4×10^{20} years to find the smallest natural number n for which it is. (The age of the universe is about 1.5×10^{10} years.) The smallest natural number n for which $991n^2 + 1$ is a perfect square is:

$$n = 12{,}055{,}735{,}790{,}331{,}359{,}447{,}442{,}538{,}767 \approx 1.2 \times 10^{28}.$$

(Check, it works.) ■

Moral. Providing a counterexample is **conclusive proof** that an implication is false, but checking even a large number of examples (unless you can exhaust all possible cases!) doesn't prove an implication in general.

1.12 Direct Proof

Remember that if A and B are predicates, the statement "If A, then B" is true if for all values of the variable(s) that make A true, B is true, also. One way to prove "If A, then B" is to check all possible cases where the hypothesis holds and see if the conclusion is also true. Of course this becomes cumbersome if the number of cases is large and impossible if it is infinite. We certainly cannot check them one by one. So we assume an abstract situation in which the hypothesis holds (nothing can be assumed beyond the hypothesis itself) and show that the conclusion must hold also.

Thinking about an example should help. Consider the statement "If x is an even integer, then x^2 is an even integer." I suspect that when you conducted the "thought experiment" you decided that this is true. It is a case in which there are infinitely many values of x that make the hypothesis true. So we will have to assume (in the abstract) that x is even and then show that x^2 has to be even, too.

When an implication is proved by assuming that the hypothesis is true and then showing that the conclusion is also, the proof is called a **direct proof.**

If we are to get anywhere, we first have to recall what it means to say that an integer is even:

Let z be an integer. Then z is said to be even *if there exists an integer w such that $z = 2w$.*

Here is the proof that if x is an even integer, then x^2 is an even integer.

Proof. Suppose that x is an even integer. Then by definition of even integer, we know that there must exist an integer y such that $x = 2y$. Now we have to show that there is an integer w so that $x^2 = 2w$. Let $w = 2y^2$. Since the product of integers is an integer, $w = 2y^2$ is an integer. Notice that

$$x^2 = (2y)(2y) = 2(2y^2) = 2w.$$

Thus x^2 is an even integer. ∎

This argument works for *any* even number; thus all cases have, in some sense, been checked.

> **The Role of Definition:** The engine that drove our argument was the *definition* of even number. The vague notion that even integers are those in the list $0, \pm 2, \pm 4, \pm 6, \pm 8, \ldots$ could not give us the power we need to prove the theorem. As I stressed in the Introduction, definitions are *tools* that we use to express abstract concepts using mathematical statements. Without careful definitions, we have nothing on which to apply the rules of logic. Insight comes from an intuitive understanding of what the terms mean, from checking examples, and so forth. But theorems are *proved* by applying logical principles to abstract definitions.

1.12.1 EXERCISE

If you haven't done so already, use a direct proof to prove that "If $x + y$ is even and $y + z$ is even, then $x + z$ is even." □

Besides the direct proof, two other methods for proving theorems are very commonly used: proof by contrapositive and proof by contradiction.

1.13 Proof by Contrapositive

1.13.1 EXERCISE

Let A and B be predicates. Construct a truth table to show that the following statements are equivalent:

$$A \Longrightarrow B \quad \text{and} \quad \sim B \Longrightarrow \sim A.$$

□

Remember, we have said that equivalent statements can be thought of as the same statement expressed in different ways. In this case, "If not B, then not A" should be viewed as a rephrasing of "If A, then B."

1.13.2 DEFINITION

The statement $\sim B \Longrightarrow \sim A$ is called the **contrapositive** of the statement $A \Longrightarrow B$.

1.13.3 EXERCISE

Find the contrapositives of the following statements. Write things in positive terms wherever possible.

1. If $x < 0$, then $x^2 > 0$.
2. If $x \neq 0$, then there exists y for which $xy = 1$.
3. If x is an even integer, then x^2 is an even integer.
4. If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.
5. If f is a polynomial of odd degree, then f has at least one real root. □

1.13.4 EXERCISE

Use your intuition about implication to explain why "If A, then B" and its contrapositive are saying the same thing. □

Sometimes it is easier to prove the contrapositive of a statement than it is to prove the statement itself. (The contrapositive gives you different statements to work with which may simply be more tractable.) Since they are equivalent, proving "If not B, then not A" is the same as proving "If A, then B." A proof in which the contrapositive is proved instead of "If A, then B" is called a **proof by contraposition** or a **proof by contrapositive.**[4]

Since the contrapositive of an implication is itself an implication, the *procedure* for doing a proof by contraposition is to figure out what the contrapositive is (as in Exercise 1.13.3) and then to proceed exactly as one does in a direct proof.

1.14 Proof by Contradiction

1.14.1 EXERCISE

Let A, B, Q, and P be statements. Construct a truth table to show that the following statements are equivalent:

$$Q \quad \text{and} \quad (\sim Q) \Longrightarrow (P \wedge \sim P).$$

[4] The latter is less grammatical, but more commonly used!

In particular, explain why this means that:

$$A \Longrightarrow B \quad \text{and} \quad (A \wedge \sim B) \Longrightarrow (P \wedge \sim P)$$

are also equivalent. (Note that logically the statement P need have no connection whatsoever with the statements A, B, or Q, though it often does in practice.) □

1.14.2 EXERCISE

To help you see why this equivalence makes sense, suppose you have statements X and Y in which

$$X := (A \Longrightarrow B), \text{ so that } \sim X := A \wedge \sim B, \text{ and}$$
$$Y := P \wedge \sim P.$$

If you know that $\sim X \Longrightarrow Y$ is true and Y is false, what can you say about the truth value of X? □

Recall that "$P \wedge \sim P$" is a contradiction, a statement that is always false. (Clearly, a statement and its negation cannot both be true.) There is a proof technique, called **proof by contradiction,** in which we first assume that the statement we want to prove is false and then show that this implies the truth of something that we know to be false. (For instance, if your reasoning ends in the conclusion that $1 = 0$, you have arrived at a contradiction since $1 = 0$ and $1 \neq 0$ cannot both be true.)

To be specific, suppose that we want to prove that $A \Longrightarrow B$ is true. We know that the negation of $A \Longrightarrow B$ is $A \wedge \sim B$. Exercise 1.14.1 tells us that if we assume $A \wedge \sim B$ and can reason our way to a contradiction, we will be able to conclude that $A \Longrightarrow B$ is true. When we do this, we are doing a **proof by contradiction.**

1.14.3 EXAMPLE (Proof by contradiction)

If $a > 0$, then $1/a > 0$. ■

Proof. The proof is by contradiction. Thus we assume the hypothesis ($a > 0$) and the negation of the conclusion ($1/a \leq 0$). Since $1/a \leq 0$, there is some nonnegative number b so that

$$1/a + b = 0$$

Multiplying both sides by a, we get

$$1 + ab = 0.$$

Since $a > 0$ and $b \geq 0$, $ab \geq 0$. Hence $1 \leq 0$. Since we also know that $1 > 0$, we get the desired contradiction. We therefore conclude that our orginal assumption must have been false, so

If $a > 0$, then $1/a > 0$

is true. ■

1.14.4 EXERCISE

Use proof by contradiction to prove that "If x is an integer, then x cannot be both even and odd." □

Remark. The problem with the preceding proofs is that each proof necessarily appealed to statements that were at least as in doubt as the statements that were being proved. (Go back and look at the proofs. Can you see where?) This is the basic problem with exercises that ask you to prove mathematical statements before any real mathematics has been discussed. They almost always amount to proving statements that we already "know" to be true using other statements that we already "know" to be true.

You should think of the proofs in this chapter as giving you practice only in using the "logical form" of the proof techniques. You should not think of them as having mathematical content. After this chapter, the proofs that you do should appeal only to the definitions and theorems that have already been discussed in a mathematically rigorous way. When you do this, you will know exactly what the starting assumptions were. You will be building a strong chain of mathematical reasoning whose beginning and end you can see. You will, thus, be standing on much firmer mathematical ground than you have been in the preceding proofs.

1.15 Proving Theorems: What Now?

In the latter part of this chapter, we have talked a bit about the logical basis for several proof techniques: direct proof, proof by contraposition, proof by contradiction. And we have talked about what general approach to take when proving existence and uniqueness theorems. But caution! You are far from being an expert on how to use these techniques; therefore, **you are not leaving this chapter!** You are just beginning to use it. In the course of working through the mathematics in the pages that lie before you, you should turn back to this chapter on a regular basis (at least for a while). When trying to decide on a strategy for proving something, review the various proof techniques and weigh them as options in your mind. Sometimes I will give you a hint as to what I would do. However, there is rarely only a single way of doing things. If my hint doesn't seem to be working for you, try something else. Proving theorems is a creative process. You may create something different from your neighbor. One proof may be shorter or more elegant or more revealing or simply more aesthetically pleasing than another, and you may want to strive for such improvements as you get more proficient. But the bottom line is that a proof is a proof is a proof. For now, concentrate on finding sound arguments that will prove the theorems you encounter. Use any and all tools at your disposal.

PROBLEMS

1. Suppose we understand the free variable z to refer to (a) books, (b) automobiles, and (c) pencils. For each context,
   - Give an example of a predicate $A(z)$ for which "For all z, $A(z)$" is a true statement.
   - Give an example of a predicate $B(z)$ for which "For all z, $B(z)$" is false but "There exists z such that $B(z)$" is true.
2. Is it possible to have a predicate $T(x)$ such that "For all x, $T(x)$" is true, but "There exists some x such that $T(x)$" is false? Justify your answer.
3. Consider the statements

   P := "Dogs eat meat."

   Q := "Rome is in Italy."

   R := "Chocolate prevents cavities."

   S := "The moon is made of green cheese."

   Determine whether each of the following is true or false.

   (a) If P, then Q. (b) If P, then R. (c) If R, then S.
   (d) If S, then Q. (e) If Q, then S.
4. With apologies to Sidney Harris for trodding on his terrific cartoon (shown in Figure 1.1), I'd like to play a little with the dog's statement. Consider the assertions made by the dog:

   A := "All cats have four legs."

   B := "I have four legs."

   C := "I am a cat."

   (Are these assertions statements or predicates? Explain.)

   The dog's statement is of the form "if A and B, then C."

   (a) Construct a truth table for the statement "If A and B, then C."

   (b) Now consider the actual truth values of the assertions made by the dog. Cross out the lines of the truth table that don't apply *in this particular instance*. What do you see?
5. This problem refers to the equivalence discussed in Example 1.7.6.

   (a) Using your intuition about implication, explain why it makes sense to say that

   If A is true, then either B is true or C is true

   means the same thing as

   If A is true and B is false, then C is true.

   (b) Go back to the "thought experiment" in Section 1.1. Find a statement that is written in the form "If A, then B or C." Find two equivalent rephrasings of the statement. (Did you intuit these rephrasings when you worked with the problem during the thought experiment?)

   (c) Construct a truth table to show that $(A \implies (B \vee C))$ *cannot* be rephrased as $((A \implies B) \vee (A \implies C))$. Using the statement you discussed in part (b) as an example, explain why.
6. Consider the statement "Marlene has brown hair." When asked to negate this statement, some students are apt to say, "Marlene has blond hair." Explain why this is incorrect. (*Hint:* There

is an important difference between a statement that is false when "Marlene has brown hair" is true, and the negation of "Marlene has brown hair.")

(For a more mathematical, but parallel, example: Ask yourself why $x = 3$ is not the negation of $x = 7$. What *is* the negation of $x = 7$?)

7. Suppose that (a, b) and (c, d) are two distinct points in $\mathbb{R}^2$. Use the processes described in Sections 1.9 and 1.10 to prove that there exists a unique line passing through the two points. (Remember that the work you do to determine the candidate in the existence part is not part of your proof.)

8. Describe what you would have to do to show that an object is *not* unique.

9. Your goal will be to prove that "If x is an odd integer, then x^2 is an odd integer."

   (a) Here are two possible definitions for an odd integer.
   - An integer z is odd if it is not even.
   - An integer z is odd if there exists an integer w such that $z = 2w + 1$.

   Which of these two definitions do you think will be more useful to you in the proof? Why?

   (b) Prove that the square of an odd integer is odd.

QUESTIONS TO PONDER

This is the first in a series of sections titled **Questions to Ponder.** In these sections you will find questions that you can play with at your leisure; you may work with other students in your class or challenge friends that are not in your class. Some of the questions should be resolvable with a bit of work. Some will become tractable as you proceed through the book. Some will be harder, and you may not be able to solve them completely, but I will only include such problems if you can make some progress on them at least by looking at examples. Some questions are philosophical in nature and their answers may be open-ended.

1. The following two statements were given as alternate definitions for an odd integer:
   - An integer z is odd if it is not even.
   - An integer z is odd if there exists an integer w so that $z = 2w + 1$.

   One would hope that these definitions are equivalent. (That one is just a rephrasing of the other!) Can you prove this?

2. You should try to prove that $\sqrt{2}$ is irrational. (*Remember:* A rational number is a number that can be written as a ratio of integers. One classic proof assumes $\sqrt{2}$ is rational; that is, $\sqrt{2} = \frac{m}{n}$. What can you say about the prime factors of m^2 and $2n^2$ and what does this tell you?)

3. Try to prove that there are infinitely many prime numbers.

4. Try to prove that every positive number has a positive square root. (That is, prove that "For any positive real number x there exists a positive real number y such that $y^2 = x$.")

5. Can every positive integer be written as the sum of distinct powers of two?

6. Can every even integer greater than 2 be expressed as the sum of two prime numbers? (This is the famous "Goldbach conjecture." The question was first asked in 1742. Mathematicians continue to struggle with it today. No one knows the answer.)