

MAT2000 Project

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The Banach–Tarski paradox and amenable groups

The Banach–Tarski paradox is a famous counter-intuitive theorem in mathematics. It states that a ball in three-dimensional space can be partitioned into finitely many pieces and re-assembled into two balls of the same size as the original one.

It turns out that the Banach–Tarski paradox is strongly related to a certain property in group theory, namely amenability. Roughly speaking, a group G is called amenable if we can write it as a countable union of subsets that do not change too much when we translate them by elements from G . In terms of amenability, the Banach–Tarski paradox is ultimately a consequence of the fact that the free group \mathbb{F}_2 is not amenable. Indeed, this was the original motivation for the definition of amenability by von Neumann in 1929. Since then, amenability has proved to be a very useful concept in several disciplines of mathematics.

The aim of this project is two-fold: The first goal is to study amenable groups, in particular give examples and non-examples and prove the equivalence of various different definitions of amenability. The second goal is to prove the Banach–Tarski paradox and explain its relation to non-amenability.

Background knowledge: Some background knowledge of group theory is essential, for instance in the form of the course MAT2200 (Groups, rings and fields). It is an advantage to have taken courses such as MAT1140 (Structures and arguments) and MAT2400 (Real analysis). I also recommend taking MAT3400/4400 (Linear analysis with applications) parallel to writing the project.

