MAT2000 project: Proving existence of solutions of nonlinear PDEs

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1. Background

As a general rule of thumb, *linear* partial differential equations (PDEs) are "easy" to solve and it is possible to find solution formulas, while *nonlinear* PDEs are difficult to solve and do not have explicit solution formulas. It might still be possible to prove that there exists a solution and that this solution has certain properties, without know what the solution is. Proving such statements is what modern PDE theory is all about.

2. Project description

The goal of this project is to read and understand the proof of existence of solutions of the nonlinear PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial^2 u}{\partial x^2} & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{1}$$

(a *convection-diffusion equation*) where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given functions. The PDE (1) is nonlinear whenever f is nonlinear—which it is in most real-world models. Such equations appear in many different applications; for instance, the Navier–Stokes equations (which are used by meteorologists to predict the weather, by oceanographers to simulate tidal waves and tsunamis, and much more) is of this type. The idea of the existence proof is to use a *fixed point iteration* where a (non-homogeneous) heat equation is solved at each iteration.

A secondary goal will be to prove properties of this solution, such as differentiability of u and "maximum principles", i.e. proving that the solution values $u(x,t)$ cannot become "too large".

As a tertiary goal, the student can try to prove existence of the initial-boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial^2 u}{\partial x^2} & x > 0, t > 0 \\ u(x,0) = g(x) & x > 0 \\ u(0,t) = h(t) & t > 0 \end{cases} \tag{2}$$

where $h(t)$ is now a prescribed function value at the boundary $x = 0$ of the domain.

3. Tools and necessary background

The project should be suitable for anyone with an interest and knowledge of real analysis, and serves as an example of how analysis techniques can be applied to tackle real-world problems.

The main tools that will be needed are Banach's fixed point theorem (see e.g. MAT2400), the fundamental solution of the heat equation (MAT3360 or MAT4301), some real analysis such as L^p spaces and the Arzela–Ascoli theorem (MAT2400 and/or MAT3400), and some PDE techniques (MAT3360 or MAT4301).

The student should have taken MAT2400, and have taken (or take in parallel) MAT3360 or MAT4301. Some relevant references for PDE theory are [1, Chapter 2.2, 2.3], [2, Appendix B], [3].

References

[1] L. C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Series in Mathematics*. American Mathematical Society, second edition, 2010.

[2] H. Holden and N. H. Risebro. *Front Tracking for Hyperbolic Conservation Laws*. Springer-Verlag Berlin Heidelberg, second edition, 2015.

[3] A. Tveito and R. Winther. *Introduction to Partial Differential Equations*, volume 25 of *Texts in Applied Mathematics*. Springer-Verlag Berlin Heidelberg, 2005.